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THE LONDON MATHEMATICAL SOCIETY.

SPEECH OF PROFESSOR DE MORGAN.

PRESIDENT,

At the First Meeting of the Society, January 16th, 1865.

I MUST remind you that to societies such as the present the maxim "Nascitur non fit" is applicable. The society should be what it will be: it will form itself. Its members will gradually adapt themselves to it and fall into their places. All similar societies take their growth from a few individuals, the most energetic who give themselves up to the management and conduct of them. If these few members can catch the general tone of those who wish to form the society, the interest will be permanent, and the undertaking successful; but if they should fall short of this, and, in carrying out their own intentions, fail to meet the views of the general body of members, there will be little hope of harmonious concert or of energetic action. There is a great deal said about the "love of science," the "love of truth," and so forth; but, in plain truth, it is a condition that people must be pleased. They will wish to further the Mathematics: very possibly—but they will wish to further them in that way which is in accordance with their own tastes and feelings. There is much discussion as to what our society should be. But this cannot be settled and marked out; it must be determined by the disposition of the members. All scientific societies are in danger of getting into a groove, and settling into a routine which possesses small interest to the great body of the members. And this may occur even when the subject is so definite as that of Geology, Botany, or Geography. And, on the other side, there is always a danger of guiding the society off the rail, as it were, and getting it out of the way in which the momentum of the members can be applied to move it.

Our subject is really rather a wide one; but there are mathematical publications in which it is contracted; and it is often

treated as a narrow subject. Our great aim is the cultivation of pure Mathematics and their most immediate applications. If we look at what takes place around us, we shall find that we have no mathematical society to look to as our guide. The Royal Society, it is true, *receives* mathematical papers, but it cannot be called a mathematical society. The Cambridge Philosophical Society seems to fulfil more nearly the functions of a mathematical society; but it is in an exceptional position. It is the society of the place which may be regarded as the centre of the mathematical world; — it is a society in which almost all the members are able to relish its highest discussions. But in London we have no mathematical society at all. There is none except the Royal Society, which may be said to contain a mathematical section.

We turn from the societies and look at the mathematical publications, and we find that they fall each into its own particular line—a line which may be useful, and which may make their volumes of lasting worth, but which is still confined and partial. The early volumes of the 'Cambridge Journal' contained papers by Duncan Gregory of the highest value, and also the earlier papers of Dr. Boole on the Calculus of Operations, and other analogous subjects. They appeared anonymously, but they gave a great impulse to the study of the particular branches of Mathematics on which they were written. But however useful such a journal may now be as a chronicle of the *past*, however fit it may still be to instruct the *future*, we must do something of a wider character for present instruction and present attraction. In the existing mathematical journals and publications, the offered inducements are, for the most part, confined to one particular class of subjects. The 'Quarterly Journal of Mathematics,' which succeeded the 'Cambridge Journal,' is almost wholly given to a particular branch of subjects; a most excellent branch it may be, but still many are repelled by this preponderance of subjects of one particular kind.

As in a journal, so it might be here; if the writers be few and their subjects confined, the supporters of the society will be few.

There was another mathematical journal which I should wish to notice; and we may call it the *Woolwich Journal of Mathematics*, as most of its contributors were men connected with the Military Academy. I find, with regard to it, that it perished

under a still more special regimen, as it came to be nothing more than an analysis of the properties of the Conic Sections and curves of the second order. The peculiar tastes of a small knot of contributors caused this journal to be overloaded with such matters; insomuch that when you looked into its pages, before you could find a bit connected with the Differential or Integral Calculus, you would come across half-a-dozen applications or proofs of Pascal's Theorem. This will be our tendency also if we do not take care; and what I particularly desire is to see this Society having *the great bulk of its business adapted to the great bulk of its members*. In order to do this, those versed in the higher Mathematics, in Canonical Algebra, Elliptic Transcendentals, and a dozen other such subjects which have been opened lately, must study the inclinations and pursuits of those engaged on the more elementary branches. All these subjects are thriving in their own way, and are laying the foundation of something great to be discovered by and by. Much as we plume ourselves on the recent development of the Mathematics, we are in a position which we can best estimate by looking at the growth of the differential and integral principles before they were formed into a regular Calculus. As it is now in our case, so it was then. There were a great number of different lines of investigation, which were apparently as isolated as those that I have mentioned as being recently developed. In the seventeenth century there was a great body of mathematical work, consisting of detached investigations and problems. None but students of history know that there is not a single branch of these two Calculi of which we should not find specimens in the works which were written at that time. In them we find integration, the solution of Differential Equations—nay, even their transformation from one system of coordinates to another, and also a number of the applications of these equations. There is, in fact, nothing of which there was not an instance at that time. If we open Wallis's book on the Centre of Gravity (now quite forgotten and unknown), we shall find an enormous body of separated problems. It would be supposed at that time that Wallis was giving a new branch to Mathematics; and was striking out an entirely new line. The whole of it merged at once, however, in the Integral Calculus, when that was formed and reduced to shape; and mathematicians now do with a few strokes of the pen

what it formerly took at least three or four pages of Geometry to effect.

Much the same kind of thing doubtless is going on now. We may be able to take our part in promoting what will be some of these days a new Calculus; though what it will be, what form it will assume, we can say no more than Wallis could have pointed out the system to which his investigations were tending. Neither he nor any one else of his time could have said what it was that was destined to serve as the connecting link between Napier's discoveries with regard to the continuous variation of arithmetical and geometrical progressions, Wright's meridional parts and the tables which he formed, and the mass of speculations of Wallis, Des Cartes, Barrow, and many others whose names I might mention.

We must endeavour, therefore, to do our part in the additions to these branches of the science, but at the same time we must take care not to let any one particular branch overgrow us.

Now, the English mathematical world of the present day takes its tone principally from the Cambridge examinations; there is no doubt of that, and there is no use in denying it. The Cambridge examination is nothing but a hard trial of what we must call problems — since they call them so — between the Senior Wrangler that is to be of this present January, and the Senior Wrangler of some three or four years ago. The whole object seems to be to produce problems—or, as I should prefer to call them, hard ten-minute conundrums. These problems, as they are called, are, and are necessarily obliged to be, things of ten minutes or a quarter of an hour. It is impossible in such an examination to propose a matter that would take a competent mathematician two or three hours to solve, and for the consideration of which it would be necessary for him to draw his materials from different quarters, and see how he can put together his previous knowledge, so as to bring it to bear most effectually on this particular subject. It is, I say, impossible that such a problem as this should be set in these examinations.

It must be one of our objects to introduce into our own discussions something more like problems, properly so called, and, if possible, to keep ourselves from entertaining an undue number of the questions just described. In some quarters the Mathematics are looked at, I may say almost entirely, with reference

to their applications. These applications are not only physical applications or commercial applications, which may be termed *external;* but there are also what are rather *internal* applications. Those very questions, to which I have alluded already, respecting curves of the second order, are applications of the principles of pure Mathematics. They form, in fact, a particular branch of the application of first principles.

We must not mistake and misapprehend these internal applications—we must not regard them as constituting entirely what we are to turn our attention to. We have several things before us, besides these, which are very little attended to. One of these is what may be called Logical Mathematics. We want a great deal of study of the connexion of Logic and Mathematics. Where is any consideration of this question to be found? If I may be allowed to say more on a subject to which I have devoted a good deal of time and thought, I would make a few remarks on this very important and yet very much neglected subject.

There is exact science in two branches: the Analysis of the necessary *Laws* of Thought, and the Analysis of the necessary *Matter* of Thought. The necessary matter of thought, that without which we cannot think, consists of Space and Time. These exist everywhere, and we can imagine no thought without them. Space and Time are the *only* necessary matters of thought. These form the subject matter of the Mathematics. The consideration of the necessary *Laws* of thought, on the other hand, constitutes Logic. These latter have been little studied hitherto, even apart from the study of the necessary matter of thought. The connexion of the two is a subject which I hope will be very seriously attended to. The taste for these investigations is one which ought to be promoted in every possible way. We mathematicians may very easily improve our reasoning from the very beginning. For, though the logic that Euclid used is very accurate, there has been no inquiry made with regard to it; and the consequence is that for two thousand years we have been proving, as we go through the elements of Geometry, that a thing is itself. That is to say, we have been proving, in the Elements of Geometry, by help of a syllogism, a thing which must be admitted before syllogism itself can be allowed to be valid. Thus, does Euclid not prove that, when there is but one A and but one B, if the A be the B, then the B is the A? He would not take such a thing

as that without appearance of proof. "A thing is itself"; that is the assertion, that is what Euclid would not take without proof!

To take an example. Let us suppose that there is a village which contains but one grocer and but one Post-Office. Then, if the grocer's be the Post Office, the Post Office is the grocer's. For, if it be possible, let the Post Office be somewhere else, say at the chandler's. Then, because the Post Office is the chandler's and the grocer's is the Post Office, it follows that the grocer's is the chandler's, another place, by hypothesis; which is absurd; and so with every other place except the grocer's. Therefore the Post Office is at the grocer's.

Is not this mode of proof in the Third Book of Euclid, being the way in which Proposition 19 is deduced from Proposition 18? Yet anybody who should use it out of geometry would be laughed at, though Euclid used it, and all those who have studied his Elements have been proving things in this manner for two thousand years. This is admitted by those who would defend Euclid from this and all similar attacks; but, say they, though he did require proof of everything, it was only from a desire to take nothing for granted. But can we say that this is only a superfluity of proof, a thing not to be regretted; which is no great harm, and which only shows that Euclid was excessively cautious? It is more than this. The proposition is as follows:—

If there be but one Post Office and but one Grocer;
And if the Grocer's be the Post Office; then
The Post Office is the Grocer's.

The proof contains a syllogism: that is, I say that by means of a syllogism is proved that which must be admitted before the syllogism itself is valid. For how do you know that if

C is B and
A is C,
A is B,

if you do not know that one A is the same as the other A, and that one B is the other B, &c.? There is no doubt about the matter, I say — and it will appear more distinctly on further thought—that you are proving by help of a syllogism what must be admitted before syllogism itself is valid. These are things

which come into Mathematics—even into Euclid—and constantly bring us to the verge of bad reasoning.

Look at the Second Book of Euclid, and turn to the Proposition : If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts ;* into the proof of which he introduces the proof that the parallelograms about the diagonal of a square are squares. This is the very next thing to bad reasoning. It is the introduction of what has nothing whatever to do with the proposition to be proved.

There are many similar points that might be noticed, and we shall find throughout the Logic of Mathematics a very great deal to think of and to discuss.

The History of Mathematics is another thing that is unfairly neglected. I say that no art or science is a liberal art or a liberal science unless it be studied in connexion with the mind of man in past times. It is astonishing how strangely mathematicians talk of the Mathematics, because they do not know the history of their subject. By asserting what they conceive to be facts they distort its history in this manner. There is in the idea of every one some particular sequence of propositions, which he has in his own mind, and he imagines that that sequence exists in history ; that his own order is the historical order in which the propositions have been successively evolved. The mathematician needs to know what the course of invention has been in the different branches of Mathematics ; he wants to see Newton bringing out and evolving the Binomial Theorem by suggestion of the higher theorem which Wallis had already given. If he be to have his own researches guided in the way which will best lead him to success, he must have seen the curious ways in which the lower proposition has constantly been evolved from the higher.

When I was a student at Cambridge I used to take walks with a friend, and we used to discuss different points in Mathematics. On one of these occasions my friend remarked, in connexion with a question of Logarithms, that he doubted whether Napier had been so great a man as he was accounted. His system, he said,

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* Prop. IV.

was nothing but a very natural deduction from the exponential symbol (a^x). Given this, the symbol follows immediately. But he was not aware, nor I either, that in all time these inversions have been made with great difficulty, and that Napier in particular did not know anything of exponents. No approach to such a symbol as the general exponent was made even by Stevinus and the other writers of his day. Napier's Algebra has recently turned up, and if ever there were a man whose ideas of powers were limited to the second power, the third power, &c. —and even these simple exponents expressed by the most complicated combinations of letters—if there ever were such a man as this, I say, it was Napier. He seems to have had less of the idea of exponents than any one else of his time. Instead of the invention of the exponential symbol giving rise to Logarithms, the reverse is more nearly true.

The history of Mathematics involves a great number of curious points, and it has not as yet been very minutely investigated. A little while ago I looked at an old German book of Arithmetic which was published in the year 1489. In it the signs for plus and minus (+ −) frequently occur, thus showing that they were used long before what is usually supposed. These signs must have been invented before that time, since the book contains the use of them fully and very frequently. We have now to look back and see whether these signs of (+) and (−) were invented by mathematicians at all. It is possible that they were originally marks used in the warehouse, and did not signify addition and subtraction. They meant, as their names import, *more* and *less*.

The first problem in which these signs were used was in a case of this sort:—There are a number of wooden chests of which the prices are to be found; and the weights of which are given. They are added up in this way. (The weight employed is a *centner*, or 100 pounds.) When a chest was about 3 centners, more or less, they put the three centners into one scale and the chest into the other. The chest was found to be a few pounds more or less. They put the few pounds either into one scale or the other accordingly, in order to make them balance. If the chest were of less weight they put them into the scale in which the goods were. Then the weight of the chest would be written thus: 3—15 lb., say: There are 3 centners and 15 pounds

less. So in other place the chest might be 3 centners and 22 pounds more, which would be written thus: 3+22. There are 14 or 15 of these written in a column. Perhaps for convenience the extra weights were put on the chests themselves. But how it was that the mathematician saw them I do not know. Could we enter into the commercial records of the time we might be able to learn.

The question arises, however, why may not the mathematicians have made these signs? The answer is, that they were not apt at the making of symbols. In order to prove this, we must look at their minds. There is an instance in this very book to which I have been referring—a work on arithmetic by John Widmann. There we find laid down a notation for the rule of three in fractions, which was used for a whole century, and which occurs, amongst other books, in the books of Tartaglia and Pacioli. Instead of

$$\tfrac{3}{4} : \tfrac{7}{11} :: \tfrac{2}{19} \quad \text{we find} \quad \tfrac{3}{4} \times \tfrac{7}{11} = \tfrac{2}{19}.$$

The lines denote that in the first instance the multiplication was to take place between numerator and denominator and denominator and numerator; in the 2nd, that numerator was to be multiplied by numerator and denominator by denominator. They put these marks and they ran them through a succession of books. For one whole century this was the canon of notation—from 1489 downwards. If this had happened in our day, and we had been in want of signs for multiplication and division, would not = and × have been used? They would have been two very apposite signs for the purpose. A schoolboy in our day would have thus separated them, if he had been accustomed to the whole. We know very well that *they* did no such thing. The one which would have been used for division serves as our sign for multiplication, and the other is our sign for equality! We have got them in quite another way; and those very marks have become the symbols of multiplication and equality.

The early study of the mind of men with regard to Mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of Mathematics.

We have also questions of Language. If we do not attend to extension of language, we are shut in and confined by it. Of this Euclid is a good example. He was stunted by want of extension.

When we come to study language in connexion with Logic, we find a great many things which would hardly have been expected, and by which we may learn how we may best extend the meanings of our terms. Would any one suppose, at first sight, that "of" and "but" are connected together in the manner of positive and negative? "All of men," or "all men," including everything that is in the term "man." "All but man" including all that is not in the term "man." Thus, when we are speaking of animals, the first phrase means all men; the second, all animals except men. "But" thus contains the absolute opposite of "of." We find that so it is in our actual every-day meanings of the words. "All but man" is a common form of expression; and "all of men" is an easy English use of the genitive. When we begin to see the like of such things as are here pointed out, we begin for the first time to have a rational power of extending the meanings of words.

There is still one more great branch of the subject, which lies in the sphere of a Mathematical Society, and which forms part of the especial province of one which would attempt to improve elementary Mathematics. It is one great branch of the subject to look at modes of proof, and to see how much can be done easily, which usually requires a very considerable force of symbols; long rows of symbols of the Differential and Integral Calculi, complicated operations and complicated modes of thought.

Let the following be the question asked: I want to know whether there be a curve of this kind. Let the tangent to it be drawn at any point P, meeting the ordinate at the point T. Then from T draw a perpendicular to the ordinate meeting the curve at the point Q. Is there a curve such that PT = the arc PQ, for all possible positions of P? We can see by common sense that the curve can be nothing but a straight line, in which case the arc and tangent are identical, and PT is both arc and tangent. Draw a tangent to another point of the curve P′ at an infinitely small distance from P. Let this tangent meet the abscissa at T′ and let Q′ be the point at which the perpendicular to the abscissa at the point T′ meets the curve. Then at one end, the augmentations of arc and tangent are the same. But at the other end, there is one triangle in which we have the diminution of the tangent for the base and an increment of the abscissa for the hypothe-

nuse ; and another triangle in which we have the same increment of the abscissa for a base and the alteration of the arc for the hypothenuse. We see, therefore, that to comply with the conditions of the problem, the product of two cosines must be greater than unity. If this had been worked out in the usual way, it would have taken a quantity of dark symbolic process to enable us to arrive at the same result, on the supposition that the Differential Calculus had been used : whereas here we see it at once.

Here, again, is a very large field open for investigation and discussion ; namely, the bringing down of the subject from the altitudes of calculation and mechanical work to plain and simple considerations, which in their turn very often set people off on yet higher problems.

Now, here are a few points on which a great deal has yet to be done. The higher Mathematics may be carried on with much greater effect by elementary students if they will but study points of Logic, History, Language, and Perception of propositions by simple common sense. Mathematics is becoming too much of a machinery ; and this is more especially the case with reference to the elementary students. They put the data of the problems into a mill and expect the result to come out ready ground at the other end. An operation which bears a close resemblance to that of putting in hemp seed at one end of a machine and taking out ruffled shirts ready for use at the other end. This mode is undoubtedly exceedingly effective in producing results, but it is certainly not so in teaching the mind and in exercising thought. If it should chance that we find a disposition among the members of this Society to leave the beaten track and cut out fresh paths, or mend the old ones, we may make this Society exceedingly useful. But if not—if it be our fate only to become problem makers and problem solvers—there is no harm done ; we shall but add one more association to the list of journals, colleges, &c., devoted to this object. The only objection is that this branch of the subject is sufficiently well appreciated and more than sufficiently well practised already.

SYLLABUS OF LECTURE

DELIVERED AT

KING'S COLLEGE, LONDON,

JUNE 28, 1865,

ON AN

ELEMENTARY PROOF AND GENERALIZATION OF SIR ISAAC NEWTON'S HITHERTO UNDEMONSTRATED RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.[2]

BY

J. J. SYLVESTER, F.R.S.,

CORRESPONDENT OF THE INSTITUTE OF FRANCE.

THIRD ISSUE, WITH CORRECTIONS, FOR THE USE OF THE MATHEMATICAL SOCIETY OF LONDON.

LET $fx = 0$ be an algebraical equation of degree n.

Suppose $fx = a_0x^n + na_1x^{n-1} + n\frac{n-1}{2}a_2x^{n-2} + \ldots + na_{n-1}x + a_n$

[a_0, a_1, a_2, ..., a_n may be termed the simple elements of fx].

Suppose

$A_0 = a_0^2$, $A_1 = a_1^2 - a_0a_2$, $A_2 = a_2^2 - a_1a_3 \ldots A_{n-1} = a_{n-1}^2 - a_{n-2}a_n$, $A_n = a_n^2$

[A_0, A_1, A_2, ..., A_n may be termed the quadratic elements of fx].

a_r, a_{r+1} is a succession of simple elements, and A_r, A_{r+1} of quadratic elements.

$\left.\begin{matrix} a_r \\ A_r \end{matrix}\right\}$ is an *associated* couple of elements;

$\left.\begin{matrix} a_r\, a_{r+1} \\ A_r A_{r+1} \end{matrix}\right\}$ is an associated couple of *successions*.

A succession may contain a permanence or a variation of signs, and will be termed for brevity a permanence or variation, as the case may be. Each succession in an associated couple may be respectively

[1] The substance of this Lecture was communicated to the Mathematical Society of London (Professor De Morgan in the Chair), June 19, 1865.

[2] In Chap. 2 of Part 2 of the "Arithmetica Universalis," entitled "De Forma Aequationis."

a *permanence* or a *variation*. Thus an associated couple may consist of two permanences or two variations, or a superior permanence and inferior variation, or an inferior permanence and superior variation; these may be denoted respectively by the symbols $p\mathrm{P}$, $v\mathrm{V}$, $p\mathrm{V}$, $v\mathrm{P}$, and termed *double* permanences, *double* variations, permanence variations, variation permanences. The meaning of the simple symbols p, v, P, V speaks for itself.

Newton's rule in its complete form may be stated as follows:—On writing the complete series of quadratic under the complete series of simple elements of fx in their natural order, the number of double permanences in the associated series, or pairs of progressions so formed, is a superior limit to the number of negative roots, and the number of variation permanences in the same is a superior limit to the number of positive roots in fx.

Thus the number of negative roots $= \text{or} < \Sigma p\mathrm{P}$
„ „ positive roots $= \text{or} < \Sigma v\mathrm{P}$

} This is the Complete Rule as given in other terms by Newton.

The rule for negative roots is deducible from that for positive, by changing x into $-x$.

As a corollary, the total number of real roots $= \text{or} < \Sigma p\mathrm{P} + \Sigma v\mathrm{P}$, *i.e.*, $= \text{or} < \Sigma\mathrm{P}$.

Hence, the number of imaginary roots $= \text{or} > n - \Sigma\mathrm{P}$, *i.e.*, $= \text{or} > \Sigma\mathrm{V}$.

This is Newton's incomplete rule, or *first part* of complete rule, the rule as stated by every author whom the Lecturer has consulted except Newton himself.[3]

By a group of negative signs, or a negative group, if we understand a sequence of negative signs, with no positive sign intervening, this incomplete rule may be stated otherwise, as follows:—

The number of imaginary roots of an algebraic function cannot be less than the number of negative groups in the complete series of its quadratic elements.

Arithmetical illustrations:—

Relation of Newton's complete rule to rule of Descartes. Newton's "Imaginary positives," "Imaginary negatives," equivalent to $\Sigma p\mathrm{V}, \Sigma v\mathrm{V}$.

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[3] In some cases this rule appears to give very little information. *Ex. gr.*: If we take the equation $x^3 + 3qx + r = 0$, its immediate application will only show that if all the roots are real $q < 0$. If, however, we make $y = x^\lambda$, and apply the criteria to the transformed equation in y, giving λ successive values between 1 and ∞, the rule will, in fact, lead (though by a difficult process) to the true discriminative criterion; this example serves to show that there is a deeper significance seated in the Newtonian criteria than does at first sight appear.

Newton's complete rule may be made to undergo its first step of generalization as follows:—

Let the two series of simple and quadratic elements of $f(x+\lambda)$ be formed, and the double permanences due to this transformation, say, $\Sigma p\mathrm{P}(\lambda)$, or, more briefly, $p\mathrm{P}(\lambda)$, be called the number of double permanences *proper* to λ, and in like manner $p\mathrm{P}(\mu)$ the number of the same *proper* to μ.

[N.B. $p\mathrm{P}(0)$ becomes the notation for what has been termed above $\Sigma p\mathrm{P}$.]

Then we have the theorem following:—

Supposing $\mu > \lambda$, $\quad p\mathrm{P}(\mu) = \text{or} > p\mathrm{P}(\lambda)$,

or, more exactly, $\quad p\mathrm{P}(\mu) - p\mathrm{P}(\lambda) = (\mu, \lambda) + 2k$,

where (μ, λ) denotes the number of real roots included between μ and λ, and k is zero or any positive integer.

Statement of this theorem in general terms.

This is to Newton's what Fourier's is to Descartes's.

Fourier's theorem recalled; it may be stated as follows:—

Form the *simple* elements appertaining to $f(x+\lambda)$ and to $(x+\mu)$; then

$p(\mu) - p(\lambda) = (\mu, \lambda) + 2k$, briefly $p(\mu) - p(\lambda) = \text{or} > (\mu, \lambda)$; as a consequence

$p(0) - p(-\infty) = p(0)$ for $p(-\infty) = 0$; hence permanences in fx = superior limit to number of negative roots;

$p(\infty) - p(0) = v(0)$ for $p(\infty) = n$; hence variations in fx = superior limit to number of positive roots.

So from the new theorem, briefly No. 1 Theorem (presently to be established in a more general form), viz.:

$$p\mathrm{P}(\mu) - p\mathrm{P}(\lambda) = \text{or} > (\mu, \lambda),$$

since $\quad p\mathrm{P}(0) - p\mathrm{P}(-\infty) = p\mathrm{P}(0)$;

for $\quad p(-\infty) = 0, \quad p\mathrm{P}(-\infty) = 0$;

we obtain $\quad p\mathrm{P}(0) = \text{or} > (0, -\infty)$.

On examination it will be found that $\mathrm{P}(+\infty) = \mathrm{P}(-\infty) = n$ or $(n-2)$ according as the second quadratic element in $\mathrm{P}(0)$ is positive or negative.[4] Thus

$$p\mathrm{P}(\infty) - p\mathrm{P}(0) = n - p\mathrm{P}(0) = v\mathrm{V}(0) + v\mathrm{P}(0) + p\mathrm{V}(0) = \text{or} < (\infty, 0),$$

or else

$$p\mathrm{P}(\infty) - p\mathrm{P}(0) = n - 2 - p\mathrm{P}(0) = v\mathrm{V}(0) + v\mathrm{P}(0) + p(\mathrm{V}0) - 2 = \text{or} < (\infty, 0),$$

[4] If $fx = (a_0, a_1, a_2, \ldots, a\overline{\backslash}x, 1)^n$, and if $\mathrm{B} = a_1^2 - a_0 a_2$, and $h = \pm\infty$, the series of quadratic elements (when in each term only the highest power of h is retained) will be found to be a_0^2; $\mathrm{B}h^2$; $\mathrm{B}h^4$; ...; $\mathrm{B}h^{2n-2}$; $a_n^2 h^{2n}$.

which is not what is wanted; but by changing x into $-x$, and thereby commuting the variations and permanences of the simple elements of $f(x)$ one into the other, the rule for the negative roots gives the rule required, viz.: $vP(0) = \text{or} > (\infty, 0)$.[5]

No. 1 theorem may be transformed, or, rather, otherwise stated, as follows:—

The simple elements of $f(x+\lambda)$ are in fact

$$\frac{f^n\lambda}{1.2...n};\ \frac{1}{n}\frac{f^{n-1}\lambda}{1.2...(n-1)};\ \frac{1.2}{n(n-1)}\frac{f^{n-2}\lambda}{1.2...(n-2)};\ \ldots\ldots;\ \frac{1}{n}\frac{f'\lambda}{1};\ f\lambda.$$

It will not affect the successions either in this series itself or the derived series of quadratic elements if we reject the common factor $\frac{1}{\Pi(n)}$ from each term. It then becomes

$f^n\lambda; f^{n-1}\lambda; 1.2f^{n-2}\lambda; 1.2.3f^{n-3}\lambda; 1.2.3.4f^{n-4}\lambda; \ldots; 1.2...nf(\lambda)$; &c...(A); and similarly rejecting the positive factors 1^2, (1.2^2), $(1.2.3)^2$, &c., from the second, third, fourth derived terms respectively, the derived series may be written $G_n\lambda$; $G_{n-1}(\lambda)$; $G_{n-2}(\lambda)$; .....; $G(\lambda)$ ..... (B), where, in general, $G_r\lambda = (f^r\lambda)^2 - \gamma_r f^{r-1}\lambda . f^{r+1}\lambda$,

γ_r denoting the fraction $\frac{n-r+1}{n-r}$.

Theorem No. 2 stated.—If γ_r in the above association of series $\left\{\begin{matrix}A\\B\end{matrix}\right\}$ be subject to satisfy the equation in differences $2-\gamma_r = \frac{1}{\gamma_{r+1}}$, provided γ_r remains always positive from $r = n$ to $r = 1$ inclusive,[6] and if we call the number of double permanences in $\left\{\begin{matrix}A\\B\end{matrix}\right\}$, $pP(\lambda)$, then it is still true that $pP(\mu) - pP(\lambda) = (\mu, \lambda) + 2k$. The theorem will be subsequently simplified by integrating the above equation, and will be shown to include theorem No. 1.

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[5] So, in general, $vP(\lambda) - vP(\mu) = (\mu, \lambda) + 2k'$, where k' is zero or a positive integer. We have thus a second theorem as general as the first, and the two will give different limits unless $k = k'$, *i.e.*, unless $P(\mu) = P(\lambda)$, for $2(k-k') = P(\mu) - P(\lambda)$. There is nothing corresponding to this in Fourier's theorem; for the two inequalities $p(\mu) - p(\lambda) = \text{or} < (\mu, \lambda)$, $v(\lambda) - v(\mu) = \text{or} < (\mu, \lambda)$ constitute not distinct but identical assertions.

[6] As regards the necessity of the extreme limits n and 1, observe that, unless γ_n were made positive, the product of $\begin{bmatrix} f^{r+1}\epsilon & f^r\epsilon \\ G_{r+1}\epsilon & G_r\epsilon \end{bmatrix}$ [see C, p. 5] would not follow the sign of ϵ for the case of $r = n-1$, $G_{n-1} = 0$; and unless γ_1 were made positive, $f''.f$ would not be positive (see p. 7) when $G_1 = 0$; consequently the three final associated pairs of signs as x increases might pass through the successive phases $\left|\begin{matrix}+--\\+++\end{matrix}\right|\left|\begin{matrix}+--\\+0+\end{matrix}\right|\left|\begin{matrix}+--\\+-+\end{matrix}\right|$, and thus a double permanence would be *lost* in the ascending transit.

A theorem No. 3 exists, derived from an order of considerations into which this Lecture will not enter. It gives much greater generality to the value of γ_r by the introduction of a second arbitrary parameter into the criteria; see footnote ([18]).

I proceed to establish theorem No. 2 by a method precisely analogous to that used in establishing Fourier's simpler one.

For brevity, by f^r understand $f^r x$; by $f^r(\pm\epsilon)$, $f^r(x\pm\epsilon)$; and so by G_r, $G_r(\pm\epsilon)$ understand $G_r x$, $G_r(x\pm\epsilon)$.

ϵ will denote an infinitesimal.

When $f^r = 0$, $f^r\epsilon = \epsilon f^{r+1}$

When $f^r = 0$, $f^{r+1} = 0 \ldots, f^{r+i-1} = 0$, then $f^r\epsilon = \dfrac{\epsilon}{1.2\ldots i} f^{r+i}$ $\Bigg\}$.

When $G_r = 0$, $G_r(\epsilon) = \epsilon \dfrac{d}{dx} G_r$.

But
$$(f^r)^2 - \gamma_r (f^{r-1})(f^{r+1}) = 0,$$

and
$$\frac{d}{dx} G_r = (2-\gamma_r) f^r f^{r+1} - \gamma_r f^{r-1} f^{r+2}$$
$$= (2-\gamma_r) \frac{f^r}{f^{r+1}} \left\{ (f^{r+1})^2 - \frac{\gamma_r}{2-\gamma_r} \cdot \frac{f^{r-1} \cdot f^{r+1}}{f^r} f^{r+2} \right\}$$
$$= \frac{f^r}{\gamma_{r+1} f^{r+1}} \left\{ (f^{r+1})^2 - \gamma_{r+1} f^r f^{r+2} \right\},$$

or
$$G_r(\epsilon) = \frac{\epsilon}{\gamma_{r+1}} \cdot \frac{f_r}{f_{r+1}} G_{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}.$$[7]

Similarly, when $G_r = 0$, $G_{r+1} = 0$, $\ldots$, $G_{r+i-1} = 0$,
it may be proved that
$$G_r(\epsilon) = \frac{\epsilon^i}{\Pi(i)\, \gamma_{r+1} \gamma_{r+2} \cdots \gamma_{r+i}} \frac{f^r}{f^{r+i}} G_{r+i}.$$[8]

We can now trace the law of the change in the number of double permanences in the associated pair of series,

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[7] Consequently, if we write $\begin{bmatrix} f^{r+1}\epsilon & f^r\epsilon \\ G_{r+1}\epsilon & G_r\epsilon \end{bmatrix}$, the product of these four quantities obeys the sign of ϵ.

[8] *Ex. gr.*: If $G_r = 0$ and $G_{r+1} = 0$, we have

$\dfrac{dG_r}{dx} = 0$, and $\dfrac{d^2G_r}{dx^2} = (2-2\gamma_r) f^r f^{r+2} + (2-\gamma_r)(f^{r+1})^2 - \gamma_r f^{r-1} f^{r+3}$
$= (2-2\gamma_r + 2\gamma_{r+1} - \gamma_r \cdot \gamma_{r+1}) f^r f^{r+2} - \gamma_r f^{r-1} f^{r+3}$.

But
$$2-2\gamma_r + 2\gamma_{r+1} - \gamma_r \cdot \gamma_{r+1} = 3 - 2\gamma_r = \frac{2}{\gamma_{r+1}} - 1 = \frac{1}{\gamma_{r+1} \cdot \gamma_{r+2}}.$$

Thus
$$\frac{d^2G_r}{dx^2} = \frac{1}{\gamma_{r+1} \cdot \gamma_{r+2}} \left\{ f^r f^{r+2} - \gamma_r \gamma_{r+1} \gamma_{r+2} f^{r-1} \cdot f^{r+3} \right\}$$
$$= \frac{f^r}{\gamma_{r+1} \gamma_{r+2} f^{r+2}} \left\{ (f^{r+2})^2 - \gamma_{r+2} f^{r+1} \cdot f^{r+3} \right\},$$

and consequently
$$G_r(\epsilon) = \frac{f^r}{\gamma_{r+1} \gamma_{r+2} f^{r+2}} \cdot G_{r+2} \frac{\epsilon^2}{1.2}.$$

$\left.\begin{matrix} f^n, & f^{n-1}, & f^{n-2}, & \ldots, & f^1, & f \\ G_n, & G_{n-1}, & G_{n-2}, & \ldots, & G_1, & G \end{matrix}\right\}$, where $G_r = (f^r)^2 - \gamma_r f^{r-1} f^{r+1}$

as x increases *continuously*.

No change can take place except at the instant when one or more of the terms in the inferior or superior series, or in both, simultaneously become zero.

1°. Suppose a single term in the upper series, as f^r, to become zero.

Writing down the sequence f^{r+1}, f^r, f^{r-1} in conjunction with the associated terms G_{r+1}, G_r, G_{r-1},

G_{r+1}, G_{r-1} are seen to be necessarily positive, and G_r of the contrary sign to f^{r+1}, f^{r-1}.

The number of cases depending on the signs of f^{r+1} and f^{r-1} are four, but reducible to two essentially distinct ones as below,[9]

$$\left\{\begin{matrix} + & 0 & + & | & + & 0 & - \\ + & - & + & | & + & + & + \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(D).}$$

For $(x-\epsilon)$ these become respectively

$$\begin{matrix} + & - & + & | & + & - & - \\ + & - & + & | & + & + & + \end{matrix}$$

For $(x+\epsilon)$ they become

$$\begin{matrix} + & + & + & | & + & + & - \\ + & - & + & | & + & + & + \end{matrix}$$

And there is neither gain nor loss of double permanences.

2°. Suppose a single term in the lower series to become zero. From the value of G_r in terms of f^{r+1}, f^r, f^{r-1}, it follows that the two extremes of these three must have the same sign; and the signs of f^{r+1}, f^r, G_{r+1}, G_{r-1} give rise to sixteen cases, reducible to the four following essentially distinct ones[10]:—

$$\begin{matrix} + & + & + & & + & - & + & & + & + & + & & + & - & + \\ + & 0 & + & & + & 0 & + & & + & 0 & - & & + & 0 & - \end{matrix} \quad {}^{11}$$

Immediately *after* the transit of the root, see footnote ([7]), these become of the form

$$\begin{matrix} + & + & + & & + & - & + & & + & + & + & & + & - & + \\ + & + & + & & + & - & + & & + & + & - & & + & - & - \end{matrix}$$

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[9] For the simultaneous reversal of the signs of the upper line will not affect the reasoning.

[10] For neither the reversal of the signs in the upper nor in the lower line will affect the reasoning.

[11] The number of occurrences of the combination $\begin{matrix} + & + & + \\ + & 0 & + \end{matrix}$ as x ascends from λ to μ will, except for special cases, be the value of k, and the number of occurrences of $\begin{matrix} + & - & + \\ + & 0 & + \end{matrix}$ in like manner, the value of k'; k, k' having the meanings attributed to them at footnote, p. 4.

And immediately *before* the transit they were of the form

$$\begin{array}{cccc} + & + & + & \quad + & - & + & \quad + & + & + & \quad + & - & + \\ + & - & + & \quad + & + & + & \quad + & - & - & \quad + & + & - \end{array}$$

In the first of the four cases there is a *gain* of two double permanences in ascending from $x-\epsilon$ to $x+\epsilon$; in the other three cases there is neither gain nor loss.

Thus for a single vanishing of an *intermediate* term in the upper or lower series double permanences may be gained, as x continuously *increases*, but can never be lost.

The same conclusion may be also established in a similar manner when several consecutive terms of the lower series, forming a group, vanish simultaneously, without the associated upper terms any of them vanishing; and also when *two* or more consecutive terms in the upper series vanish, which necessitates all the associated terms in the lower series also vanishing; and the law which limits the increase may be ascertained, and such increase may be shown to be always an even number.

But these cases are singular cases, and may be met at once by the consideration (equally applicable to the proof of Fourier's simpler theorem) that if two or more functions of x depending on fx vanish simultaneously, this must be by virtue of one or more relations existing between the coefficients of fx; and by giving infinitesimal variations to the coefficients, we shall leave the criteria [the two sets of terms corresponding to given limits] virtually undisturbed, and may manage that the coincidents of the transits will no longer take effect; at the same time, *in general*, the character of the roots will remain unaltered as regards the number of real and imaginary ones; so that the singular cases come under the operation of the same law as the general case. There is, however, an apparent possible exception to this reasoning, viz., when fx possesses one or more groups of equal roots, in which case an infinitesimal variation in the coefficients *may* be accompanied with a change of character in the roots, such as a passage from real equal to imaginary pairs: to meet this objection without embarrassing oneself with intricate considerations as to the signs to be given to the infinitesimal variations in order to avoid liability to such change, it is better to adopt the same kind of proof as is usual with Fourier's method, which there is no difficulty in doing by aid of the expressions given above with reference to the value of $f^n(\epsilon)$, and consequently also of $f^{n+1}(\epsilon)$, $f^{n+2}(\epsilon)$, ..., $f^{n+i-2}(\epsilon)$, when f^n, f^{n+1}, ..., f^{n+i-1} are all zeros, from which may easily be deduced also the special expressions for $G_n(\epsilon)$, $G_{n+1}(\epsilon)$, ..., $G_{n+i-2}(\epsilon)$ applicable to this case;

and again, as regards the case when a group of lower terms vanish without the associated upper ones so doing, by aid of the general expressions given for the G functions last above written. But time would not suffice for going into these details in a single lecture.[12]

We must, lastly, consider what happens when one or more of the terms at either extremity vanish.

f^n and G_n are constants, and can never vanish ; and G is a square, and essentially positive.

But suppose x to become a root of fx, so that $f=0$, then the last pair of couples in the associated series immediately before the transit will be $\begin{matrix} f' ; -\epsilon f' \\ f'^2 ; \epsilon^2 f'^2 \end{matrix}$, and immediately subsequent to the transit $\begin{matrix} f' ; \epsilon f' \\ f'^2 ; \epsilon^2 f'^2 \end{matrix}$; thus there will be one double permanence *gained* when a simple root passes over.

Suppose now x to become a root of the ith order of repetition, equivalent, *i.e.*, to i roots passed over, so that $f=0$, $f^1=0$, $f^2=0$, ..., $f^{i-1}=0$, then the last $i+1$ superior terms of the association become, immediately after the transit,

$$f^i ; \frac{\epsilon}{1} f^i ; \frac{\epsilon^2}{1.2} f^i ; \frac{\epsilon^3}{1.2.3} f^i ; \ldots\ldots, \frac{\epsilon^i}{1.2\ldots i} f^i ,$$

tantamount to $1 ; \epsilon ; \frac{\epsilon^2}{1.2} ; \frac{\epsilon^3}{1.2.3} ; \ldots\ldots ; \frac{\epsilon^i}{1.2\ldots i} ;$

and the lower terms associated therewith, rejecting the common factor f^i, and certain obviously superfluous positive numerical factors besides, become

$$1 ; \left(1-\frac{\gamma_{i-1}}{2}\right)\epsilon^2 ; \left(1-\frac{2\gamma_{i-2}}{3}\right)\epsilon^4 ; \ldots\ldots ; \left(1-\frac{(1-i)\gamma_1}{i}\right)\epsilon^{2i-2} ; \epsilon^{2i}.$$

By hypothesis, $2-\gamma_{i-1} > 0$. Hence there is obviously *one* double permanence, viz., at the first couple of pairs above written, corresponding to a value of x immediately greater than a multiple root, whereas for a value of x immediately less than such root there is no double permanence at all corresponding, for the upper series will consist exclusively of variations of sign when ϵ is changed into $-\epsilon$; as regards the other terms in the lower series, they will not necessarily be all positives, unless, *in order to meet the case of* equal *roots in fx*, we determine the value of γ_r in the equation $2-\gamma_r = \frac{1}{\gamma_{r+1}}$, subject to the condition that γ_r shall be not only positive, but also subject to the limitation $\gamma_r < \frac{i+1-r}{i-r}$ for all values of i not superior to n [r of

[12] See Appendix for summary of demonstration applicable to these hypotheses.

course being supposed less than i]. This latter limitation, however, will eventually be seen to be included in the former.[13] This being the case, it follows that the number of double permanences appertaining to the associated series

$$\begin{matrix} f^n, & f^{n-1}, & f^{n-2}, & \ldots\ldots, & f', & f \\ G_n, & G_{n-1}, & G_{n-2}, & \ldots\ldots, & G_1, & G \end{matrix}$$

will be increased by at least as many units as there are real roots. equal[14] or unequal, passed over as we ascend from λ to μ, and that the excess, if any, of the increase of such number over the number of real roots will be an even integer.

To determine the value of γ_r, make $\gamma_r = \dfrac{U_r}{U_{r+1}}$,

then $$2 - \frac{U_r}{U_{r+1}} = \frac{U_{r+2}}{U_{r+1}},$$

or $$U_{r+2} - 2U_{r+1} + U_r = 0,$$

of which the general solution is

$$U_r = A + B(r-1);$$

so that $$\gamma_r = \frac{A + B(r-1)}{A + Br}.$$

If we write $A = n$, $B = -1$, we obtain

$$\gamma_{n-1} = \frac{2}{1}, \quad \gamma_{n-2} = \frac{3}{2}, \quad \ldots\ldots, \quad \gamma_2 = \frac{n-1}{n-2}, \quad \gamma_1 = \frac{n}{n-1}.$$

These are the values of γ_r in the theorem No. 1, and they satisfy the *two* conditions above stated. For (1) γ_r is positive, and (2)

$$\gamma_r = \frac{n-r+1}{n-r} < \frac{i-r+1}{i-r}$$

for all values of $i < n$.[15]

Hence theorem No. 1 is contained in theorem No. 2, and Newton's theorem is contained in theorem No. 1, so that it is a corollary in the

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[13] It is shown, at p. 10, that $\gamma_r = \dfrac{\nu+r-1}{\nu+r}$, ν having any value not intermediate between $-n$ and 0. When ν is positive, $\gamma_r < 1 < \dfrac{i+1-r}{i-r}$.

Again, when ν is negative, let $\nu = -\nu'$, then $\nu' =$ or $> n =$ or $> i$.

Hence $\gamma_r = \dfrac{\nu'-r+1}{\nu'-r} =$ or $< \dfrac{i+1-r}{i-r}$, and the required condition is still satisfied. The sole exception is when $\nu = -n$ and $i = n$, *i.e.*, when the original Newtonian criteria are those employed, and the equation has all its roots equal to one another, for which case see footnote ‡, below.

[14] Each root repeated m times counts as m roots.

[15] The degree of fx being n, i is necessarily less than n, unless all the roots are equal, a case which may be considered excluded, as then all the Newtonian criteria become zero.

second order of derivation from our theorem No. 2. The general value of γ_r is $\frac{\nu+(r-1)}{\nu+r}$, ν being any real quantity whatever not *intermediate* between 0 and $-n$. To obtain theorem No. 1 we must make $\nu=-n$. In that case $\gamma_n=\frac{1}{0}$; and accordingly, when $G_{n-1}=0$, $G_{n-1}(\epsilon)$ by the formula at p. 5 should vanish. It will easily be found that for this peculiar value $G_{n-1}(x)$ becomes independent of x, in fact a constant.[16]

Theorem No. 2 may also be stated as follows:—

If $$c_0+c_1x+c_2x^2+\ldots\ldots+c_nx^n=f(x),$$

and ν be any real quantity not *intermediate* between 0 and $-n$,

and if $$c_0,\ \frac{c_1}{\nu},\ \frac{1.2c_2}{\nu(\nu+1)},\ \frac{1.2.3c_3}{\nu(\nu+1)(\nu+2)},\ \ldots\ldots,$$

say, $$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ldots,$$

be taken as the simple elements of fx, and

$$a_0^2,\ a_1^2-a_0a_2,\ a-a_1a_3,\ \ldots\ldots,\ a_{n-1}^2-a_{n-2}a_n,\ a_n^2,$$

say, $$A_0,\ A_1,\ A_2,\ \ldots\ldots,\ A_{n-1},\ A_n,$$

as the quadratic elements of the same; and if we understand by the (cA) association the paired series

$$\left\{\begin{matrix} c_0\ c_1\ldots\ldots\ c_n \\ A_0A_1\ldots\ldots A_n \end{matrix}\right\},$$

and if $p\mathrm{P}(\lambda)$ signifies the number of double permanences in the (cA) association corresponding to $f(x+\lambda)$, then

$$p\mathrm{P}(\mu)-p\mathrm{P}(\lambda)=(\mu,\lambda)+2k,^{17}$$

where k is zero or some positive integer.

When $\nu=-n$, this theorem becomes effectively identical with theorem No. 1 in its original form.

The existence of a theorem No. 3, including No. 2, and containing

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[16] Analogous observations apply to the other extreme or limiting case—viz., that where ν is made equal to zero—for then $\gamma_1=0$, so that $G_1=(f^1)^2$, and is always positive, so that, when f^1 becomes 0, for this particular system of γ's, the three last couples in the pair of progressions before such transit may be $\begin{matrix}+&-&+\\+&+&+\end{matrix}$, and after transit $\begin{matrix}+&+&+\\+&+&+\end{matrix}$; and thus contrary to what usually happens (see p. 6), there may on the above hypothesis be an *extra* gain of two double permanences and loss of two variation permanences, thus increasing the values of k and k' in notnote (5), which of course does not affect the validity of the theorem.

[17] And of course also $v\mathrm{P}(\lambda)-v\mathrm{P}(\mu)=(\mu,\lambda)+2k'$, where k' is zero or some positive integer. Thus there are two *independent* limits obtainable from every system of quadratic elements, one afforded by the loss or gain of double permanences, the other by the gain or loss of variation permanences. Descartes' Rule offers nothing analogous; for the gain or loss of simple variations is the same thing as the gain or loss of simple permanences.

two arbitrary parameters, becomes apparent from the consideration that $fx = 0$ will have the same *finite* roots as $Fx = 0$, where

$$Fx = \epsilon x^{n+i+j} + x^i fx + \eta,$$

ϵ, η being two infinitesimals: this brings in two parameters, which, so far as this particular method of demonstration of their existence applies, would seem to be necessarily integer, but, from the nature of the question, must obviously admit of some wider definition.[18]

The labours of all preceding writers on this subject have been confined exclusively to the *imperfect* form of Newton's theorem; nor, previously to the Lecturer's communication of his Trilogy of Algebraical Researches to the Royal Society of this year, was anything more made out of it than to show that, if *any* negative terms occur in the quadratic

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[18] If we use theorem No. 2, j may be made zero (it will be found) without any loss of generality; in fact, if it be made greater than zero, the result obtained will be included in that obtained by making it equal to zero. On the other hand, if we use theorem No. 1, retaining for i, j their general values, the result obtained will be the same as that which flows from the use of theorem No. 2, with $j = 0$, except that it will be specialized by ν being restricted to integer values only. By aid of the method above indicated, we may substitute as the values of a_0, a_1, a_2, ..., a_n in p. 10 (so far, but only so far, as relates to the total number of positive and negative roots),

$$c_0\,;\ \frac{i+1}{\nu}c_1\,;\ \frac{(i+1)(i+2)}{\nu(\nu+1)}c_2\,;\ \frac{(i+1)(i+2)(i+3)}{\nu(\nu+1)(\nu+2)}c_3\,;\ \ldots\ldots,$$

provided $\nu =$ or $> i$, or $\nu =$ or $< -n$ and $i =$ or > 0. The particular form of demonstration indicated requires i to be integer; but this restriction, I have great reason to believe, indeed have scarcely a doubt, is unnecessary and may be neglected.

$$1\,;\ \frac{\nu}{i+1}\,;\ \frac{\nu(\nu+1)}{(i+1)(i+2)}\,;\ \ldots\ldots.$$

may be termed a divestible system of coefficients; such system is a generalization of the ordinary binomial system

$$1\,;\ n\,;\ \frac{n(n-1)}{2}\,;\ \ldots\ldots.$$

If we call α, β, γ, ... any system of the like nature, and form the equation

$$\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \ldots = 0,$$

it is obvious, or at least very readily proved, that the *simple* roots of this equation must all, or all but one, be imaginary. It is not unlikely that every system α, β, γ, ... which satisfies the above condition and one or more other general conditions, may be employed as a *divestible system*. The particular system, involving two arbitrary parameters, above given, will be found to satisfy the further condition that *all* its Newtonian non-trivial criteria (the A_1, A_2, ..., A_{n-1} of p. 1) become negative. When $n = 2$ or $n = 3$, the second condition implies the first; and for these cases it is easily proved that every system of quantities which satisfies the second (and therefore the first) condition, forms a valid system of divestible factors. If we make $i = 0$ and $\nu = 1$, we learn that the equation $1 + x + x^2 + \ldots\ldots + x^n = 0$ can never have more than one real root; if we make $i = 0$ and $\nu = \infty$, we learn the same of the equation

$$1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots\ldots + \frac{x^n}{1.2\ldots n} = 0.$$

Newton's own rule would only, in these and such like examples, reveal the existence of a single pair of imaginary roots.

elements, there must be *some* imaginary roots;[19] but this becomes immediately apparent if we consider f a homogeneous function of the nth degree in x and y; for then the number of imaginary roots of $\frac{x}{y}$ or $\frac{y}{x}$ in $f(x, y)$ cannot be fewer than in $\left(\frac{d}{dx}\right)^i \left(\frac{d}{dx}\right)^j f(x, y)$, i and j being *any* two integers: making $i+j=n-2$, and giving j all values from 0 to $n-2$ in succession, it will readily be seen that Newton's criteria are the quantities which respectively serve to determine whether the quadratic *functions* obtained by these various substitutions have real or imaginary roots. If any one of them is negative, one of those derivatives has imaginary roots, and therefore the primitive function $f(x, y)$ has at least one such pair.[20]

Newton's assertion (if it is his own assertion) that his rule will in *general* give the *actual* number, and not merely a superior limit to such number of real roots, is certainly more than questionable. For if $f'x, f''x, f'''x, f''''x, f'''''x$ have respectively not more than i_1, i_2, i_3, i_4, i_5 pairs of imaginary roots, this rule *cannot* reveal the existence of more than j pairs, where j is the *least* number of the set of numbers i_1+1, i_2+1, i_3+2, i_4+2, i_5+3, ...

Geometrical illustration.

Ortae a Cartesio, quam Newtonus insuper auxit,
Doctrinae en! demum, fons et origo patent.

ATHENÆUM CLUB; 26*th June*, 1865.

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[19] *See Postscript.*

[20] In like manner, if $[a, b, c, d]$ represent the discriminant with its sign reversed of $(a, b, c, d\chi x, y)^3$, the fact of any of the quantities

$$[a, b, c, d], \quad [b, c, d, e], \quad [c, d, e, f], \quad \&c.,$$

becoming negative will imply the existence of some imaginary roots in $(a, b, c, d, e, f, \ldots\chi x, y)^n$, and so in general. One would be glad to know whether, by aid of a complete table of the discriminants of a function of the nth order and of its successive derivatives [respectively $2, 3, \ldots, (n-1)$ in number], it is possible or not to ascertain the exact number of its real roots.

When n is 4, the only dubious case arising under such table is that where the discriminant of the quartic itself is positive, but those of its two derived cubics and three derived quadratics each negative. In such case it remains to be ascertained whether it is true or not that the roots of the quartic cannot all be imaginary, or (which is here the same thing) must all of necessity be real.

It seems desirable to show *a priori* that when the roots of

$$fx = x^n + npx^{n-1} + n\frac{n-1}{2}qx^{n-2} + \ldots$$

are all real, the criteria p^2-q, q^2-pr, ... are necessarily all positive. This endoscopic method of proof I have not yet been able to complete; but I have noticed that, if $\alpha, \beta, \gamma, \delta, \ldots$ are the roots of fx, and

$$\frac{fx}{(x-\alpha)(x-\beta)} = x^{n-2} + \lambda x^{n-1} + \mu x^{n-3} + \ldots,$$

the following somewhat curious relations obtain, viz.:

$$p^2-q = \Sigma\left\{(\alpha-\beta)^2(\lambda^2-\mu)\right\}, \quad q^2-pr = \Sigma\left\{(\alpha-\beta)^2(\mu^2-\lambda\nu)\right\}, \quad \&c.$$

POSTSCRIPT.

Prof. J. R. Young contests the accuracy of the assertion at ([19]), and claims to have demonstrated Newton's rule twenty years ago. Call the equation

$$(a, b, c, d, e, \ldots \nabla x, 1)^n = 0;$$

the derived cubics are of course to a factor *près* respectively

$$(a, b, c, d \nabla x, 1)^3; \quad (b, c, d, e \nabla x, 1)^3; \quad (c, d, e, f \nabla x, 1)^3; \quad \&c.$$

It is with these cubics that Prof. Young, in his argument (if it may be called so), exclusively deals. Omitting merely superfluous observations, his ratiocination runs as follows:—"If any of these limiting cubics indicate imaginary roots when submitted to the criteria, such indications will imply imaginary roots in the proposed equation. But several indications, apparently distinct, may offer themselves in those equations which, upon closer examination, may be found to be necessarily dependent or concurrent. Distinct imaginary pairs can of course be inferred only from independent and non-concurring conditions. We have therefore to inquire how these are to be discovered in the above series of equations. And first we may remark that, since only one imaginary pair can enter into a cubic equation, it follows that whether the criterion of imaginary roots is satisfied by the three leading terms of any of the above cubics, or by the three final terms, or simultaneously by both sets of three, one imaginary pair, and one only, is implied. Hence, when both sets of three terms, furnished by any cubic, fulfil the proposed conditions, these conditions, though really independent, that is, not necessarily implied one in the other, nevertheless necessarily concur in indicating the same thing. Thus only a single imaginary pair can be inferred from any one of the limiting cubics, whether the criterion is satisfied for one set of three terms or for the two consecutively.

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"If the first set of three (that is, the leading terms in the first set) satisfy the criterion, we can immediately infer the existence of one imaginary pair. If the next set (the final terms of the same cubic) also satisfy it, the preceding condition merely recurs, and supplies no additional information. In this case the following set of three, the leading terms of the next cubic ....., furnish the same concurring condition ....., and so on, until we arrive at a set of three terms for which the condition fails, thus putting a stop to the series of

concurring indications, and preparing the way for new and distinct conditions altogether unconnected with the former. As soon as the criterion again holds, the condition being thus entirely independent of, and unconnected with, the former, must imply another and distinct imaginary pair, and so on to the end of the series." Prof. Young subsequently goes on to observe that the criteria of the successive cubics (discarding repetitions) are identical with those of the original equation, which, he says, "we now know to be so connected together that if, when proceeding from one set of three terms in the equation to the three next in order, the consecutive criteria both have place, the recurrence is to be regarded merely as a second indication of the same thing—the existence of a single imaginary pair; and that as soon as the condition fails preparation is made for a new and independent indication, and so on, until all the sets of three have been examined. Hence the indications that are really non-concurrent, and consequently the number of imaginary roots inferrible from them, may be noted" according to a rule "which is virtually the same as that of Newton."

This is the sum and substance, in a simplified form, of Dr. Young's so-called proof. "It is such stuff as dreams are made of"; and, culminating as it does in a palpable *petitio principii*, does not need a detailed refutation at the hands of the author of this Lecture. It is not by such vague rhetorical processes, but by quite a different kind of mental toil, that the truths of science are to be won, and a way opened to the inner recesses of the reason.

APPENDIX

On the Singular Cases referred to in the Text foregoing (*see p.* 7).

It may easily be shown that there are only four hypotheses admissible concerning vanishing f's and G's.

1°. f^r may vanish, but not the adjacent terms nor the associated term G_r.

2°. G_r may vanish, but not the adjacent terms nor the associated term f^r.

3°. G_r, G_{r+1}, ..., G_{r+i-1} (i being greater than unity) may vanish *without any of* the associated terms vanishing.

4°. f^r, f^{r+1}, ..., f^{r+i-1} (i being greater than unity), may vanish, and *consequently* also G_r, G_{r+1}, ..., G_{r+i-1} *all* vanish.

Hypotheses (1) and (2), which are the only cases that can happen in general, have been discussed in the text preceding.

Now consider the 3rd hypothesis.

1st. Suppose i any even number (say 4), so that $G_r = 0$, $G_{r+1} = 0$, $G_{r+2} = 0$, $G_{r+3} = 0$, then

f^{r-1}, f^{r+1}, f^{r+3} have the same sign *inter se*,

and f^r, f^{r+2}, f^{r+4} have the same sign *inter se*,

and there will be four essentially distinct cases as below, representing the partial association

$$\left\{\begin{matrix} f^{r-1}, & f^r, & f^{r+1}, & f^{r+2}, & f^{r+3}, & f^{r+1} \\ G_{r-1}, & G_r, & G_{r+1}, & G_{r+2}, & G_{r+3}, & G_{r+4} \end{matrix}\right\}$$

at the moment of x taking the critical value which causes the G's to vanish, viz.:

+ + + + + +	+ + + + + +	+ − + − + −	+ − + − + −
+ 0 0 0 0 +	+ 0 0 0 0 −	+ 0 0 0 0 +	+ 0 0 0 0 −

Immediately *after* the passage of the critical value these signs become

+ + + + + +	+ + + + + +	+ − + − + −	+ − + − + −
+ + + + + +	+ + + + + −	+ − + − + +	+ − + − + −

as appears from the expression for $G_r(\epsilon)$ in terms of G_{r+i}, f^r, f^{r+i}, given at p. 5; and immediately *before* the passage these partial associations are of the form

+ + + + + +	+ + + + + +	+ − + − + −	+ − + − + −
+ − + − + −	+ − + − + −	+ + + + + +	+ + + + + −

There will thus be a gain of four double permanences in passing upwards through the critical values in the two first cases, and neither gain nor loss in the two last.

2nd. If i is any odd number, say 3, there will still be four cases, for which respectively the signs after transit will be

+ + + + +	+ + + + +	+ − + − +	+ − + − +
+ + + + +	+ + + + −	+ − + − +	+ − + − −

and, before transit,

+ + + + +	+ + + + +	+ − + − +	+ − + − +
+ − + − +	+ − + − −	+ + + + +	+ + + + −

showing a gain in the upward passage of 4, 2, 0, 0 double permanences respectively.

Lastly, let us consider the 4th hypothesis.

1st. Suppose i is any even number (say 4), and that

$f^r = 0$, $f^{r+1} = 0$, $f^{r+2} = 0$, $f^{r+3} = 0$;

then, after the transit, the f and G associated series from $r+4$ to $r-1$ become, writing for brevity's sake ϕ for f^{r+4} and ψ for f^{r-1},

$$\phi\,;\quad \epsilon\phi\,;\quad \frac{\epsilon^2}{1.2}\phi\,;\quad \frac{\epsilon^3}{1.2.3}\phi\,;\quad \frac{\epsilon^4}{1.2.3.4}\phi\,;\quad \psi,$$

$$\phi^2\,;\quad k_1\epsilon^3\phi^2\,;\quad k_2\epsilon^4\phi^2\,;\quad k_3\epsilon^6\phi^2\,;\quad -\frac{\epsilon^3}{1.2.3}\gamma_r\phi\psi\,;\quad \psi^2,$$

where γ_r is positive, and, as follows from what is proved in footnote ([12]), k_1, k_2, k_3 also will be all positive. Thus there will be essentially only two cases, according as ϕ, ψ have the same or contrary signs; after transit, the association will be of the form

$$\left|\begin{matrix}+&+&+&+&+&+\\+&+&+&+&-&+\end{matrix}\right| \text{ or } \left|\begin{matrix}+&+&+&+&+&-\\+&+&+&+&+&+\end{matrix}\right|$$

and before transit [changing ϵ into $(-\epsilon)$] of the form

$$\left|\begin{matrix}+&-&+&-&+&+\\+&+&+&+&+&+\end{matrix}\right| \text{ or } \left|\begin{matrix}+&-&+&-&+&-\\+&+&+&+&-&+\end{matrix}\right|$$

showing a gain of two double permanences, or else of four such.

2nd. If i is any odd number, as 3, the associated series corresponding to the values of r, from $r+3$ to $r-1$, using ϕ to denote f^{r+3}, will be

$$\phi\,;\quad \epsilon\phi\,;\quad \frac{\epsilon^2}{1.2}\phi\,;\quad \frac{\epsilon^3}{1.2.3}\phi\,;\quad \psi,$$

$$\phi^2\,;\quad k_1\epsilon^2\phi^2\,;\quad k_2\epsilon^2\phi^2\,;\quad -\frac{\epsilon^2}{1.2}\gamma_r\phi\psi\,;\quad \psi^2,$$

and thus the signs after transit will be

$$\left|\begin{matrix}+&+&+&+&+\\+&+&+&-&+\end{matrix}\right| \text{ or } \left|\begin{matrix}+&+&+&+&-\\+&+&+&+&+\end{matrix}\right|$$

and before transit,

$$\left|\begin{matrix}+&-&+&-&+\\+&+&+&-&+\end{matrix}\right| \text{ or } \left|\begin{matrix}+&-&+&-&-\\+&+&+&+&+\end{matrix}\right|$$

showing a gain of two double permanences on either supposition in the ascent from $x-\epsilon$ to $x+\epsilon$. And so in general for the 3rd and 4th (the two singular) hypotheses, whatever the value of i may be, an even number of double permanences may be gained in passing upwards through a critical value of x, but none can ever be lost[21]: this completes the demonstration.

[21] The number of double permanences gained on the 3rd hypothesis will be in the 4 subcases respectively $2i$, $2i$, 0, 0 if $2i$ consecutive G's vanish, and $2i+2$, $2i$, 0, 0 if $2i+1$ of them vanish; on the 4th hypothesis, in the 2 subcases the respective numbers gained will be $2i-2$, $2i$ if $2i$ consecutive f's vanish, and $2i$, $2i$ if $2i+1$ of them vanish; this statement requires a slight modification for the particular form of the theorem No. 2 corresponding to $\nu = 0$. It may be noticed that, in an exhaustive study of this theorem, two sorts of singularities occur separately or in combination, viz., those arising out of the form of the equation and those imparted to the criteria by giving critical values to the limited arbitrary parameter.

ON THE

TRANSFORMATION OF PLANE CURVES.

By PROFESSOR CAYLEY.

Read Oct. 16, 1865.

1. The expression a "double point," or, as I shall for shortness call it, a "dp," is to be throughout understood to include a cusp: thus, if a curve has δ nodes (or double points in the restricted sense of the expression) and κ cusps, it is here regarded as having $\delta+\kappa$ dps.

2. It was remarked by Cramer, in his "Théorie des Lignes Courbes" (1750), that a curve of the order n has at most $\frac{1}{2}(n-1)(n-2)$, $=\frac{1}{2}(n^2-3n)+1$, dps.

3. For several years past it has further been known that a curve such that the coordinates $(x:y:z)$ of any point thereof are as rational and integral functions of the order n of a variable parameter θ, is a curve of the order n having this maximum number $\frac{1}{2}(n-1)(n-2)$ of dps.

4. The converse theorem is also true, viz.: in a curve of the order n, with $\frac{1}{2}(n-1)(n-2)$ dps, the co-ordinates $(x:y:z)$ of any point are as rational and integral functions of the order n of a variable parameter θ — or, somewhat less precisely, the coordinates are expressible rationally in terms of a parameter θ.

5. The foregoing theorem, as a particular case of Riemann's general theorem, to be presently referred to, dates from the year 1857; but it was first explicitly stated only last year (1864) by Clebsch, in the Paper "Ueber diejenigen ebenen Curven deren Coordinaten rationale Functionen eines Parameters sind."—Crelle, t. 64 (1864), pp. 43—63.

6. The demonstration is, in fact, very simple; it depends merely on the remark that we may, through the $\frac{1}{2}(n-1)(n-2)$ dps, and through $2n-3$ other points on the given curve of the order n, together $\frac{1}{2}(n^2+n)-2$, $=\frac{1}{2}(n-1)(n+2)-1$, points, draw a series of curves of the order $n-1$, given by an equation $U+\theta V=0$, containing an arbitrary parameter θ; any such curve intersects the given curve in the dps, each counting as two points, in the $2n-3$ points, and in *one* other point; hence, as there is only one variable point of intersection, the coordinates of this point, viz., the coordinates of an arbitrary point on the given curve, are expressible rationally in terms of the

parameter θ. The demonstration may also be effected in a similar manner by means of curves of the order $n-2$.

7. Before going further, it will be convenient to introduce the term "Deficiency," viz., a curve of the order n with $\frac{1}{2}(n-1)(n-2)-\mathrm{D}$ dps, is said to have a deficiency $= \mathrm{D}$: the foregoing theorem is that for curves with a deficiency $= 0$, the coordinates are expressible rationally in terms of a parameter θ. Since, in such a curve, the different points succeed each other in a certain definite order, viz., in the order obtained by giving to the parameter its different real values from $-\infty$ to ∞, the curve may be termed a *unicursal* curve.

8. Riemann's general theorem, as applied to plane curves, is stated, but not in its complete form, by Schwarz, in the Paper, "De superficiebus in planum explicabilibus primorum septem ordinum."—Crelle, t. 64 (1864), pp. 1—17. To complete the enunciation it is necessary to refer to page 137 of Riemann's own Paper, "Théorie der Abelschen Functionen" —Crelle, t. 54 (1857), pp. 115—155; viz., the enunciation will be:—

9. For a curve of any order with a given deficiency D, the co-ordinates may be expressed as follows:—

$\mathrm{D} = 0$, rationally in terms of a parameter θ, or, what comes to the same thing, rationally in terms of the parameters (ξ, η), connected by an equation of the form $(1, \xi)(1, \eta) = 0$.

$\mathrm{D} > 0$, rationally in terms of the parameters (ξ, η) connected by an equation of a certain form, viz.—

$\mathrm{D} = 1$, the equation is $(1, \xi)^2(1, \eta)^2 = 0$, or (what comes to the same thing) η is the square root of a quartic function of ξ.

$\mathrm{D} = 2$, the equation is $(1, \xi)^3(1, \eta)^2 = 0$, or (what comes to the same thing) η is the square root of a sextic function of ξ.

$\mathrm{D} > 2$, viz.:—

D odd, $= 2\mu-3$, the equation is $(1, \xi)^\mu(1, \eta)^\mu = 0$, and is besides such that, treating (ξ, η) as Cartesian coordinates, the curve thereby represented has $(\mu-2)^2$ dps.

D even, $= 2\mu-2$, the equation is $(1, \xi)^\mu(1, \eta)^\mu = 0$, and is besides such that, treating (ξ, η) as Cartesian coordinates, the curve thereby represented has $(\mu-1)(\mu-3)$ dps.

10. To see more clearly the meaning of this, write $\frac{\xi}{\eta}, \frac{\eta}{\zeta}$ in place of (ξ, η), so that the coordinates $(x : y : z)$ are expressible rationally and homogeneously in terms of (ξ, η, ζ), connected by an equation of the form $(\zeta, \xi)^\mu(\zeta, \eta)^\mu = 0$. Such an equation, treating therein (ξ, η, ζ) as co-ordinates, belongs to a curve of the order 2μ, with a μ-tuple point at $(\xi = 0, \zeta = 0)$, a μ-tuple point at $(\eta = 0, \zeta = 0)$, and which has besides $(\mu-2)^2$ or $(\mu-1)(\mu-3)$ dps, according as $\mathrm{D} = 2\mu-3$ or $2\mu-2$. The co-ordinates $(x : y : z)$ of a point of the given curve are expressible

rationally in terms of the co-ordinates $(\xi : \eta : \zeta)$ of a point on the new curve; and we may say that the original curve is, by means of the equations which give $(x : y : z)$ in terms of $(\xi : \eta : \zeta)$, *transformed* into the new curve.

11. A curve of the order 2μ may have $\frac{1}{2}(2\mu-1)(2\mu-2)$, $= 2\mu^2-3\mu+1$ dps; hence in the new curve, observing that the μ-tuple points each count for $\frac{1}{2}(\mu^2-\mu)$ dps, we have—

In the case $D = 2\mu-3$,	In the case $D = 2\mu-2$,
Deficiency $= 2\mu^2-3\mu+1$	Deficiency $= 2\mu^2-3\mu+1$
$-\mu^2+\mu$	$-\mu^2+\mu$
$-\mu^2+4\mu-4$	$-\mu^2+4\mu-3$
$= 2\mu-3 = D$	$= 2\mu-2, = D.$

Moreover, for $D = 0$, the transformed curve is a conic, with 0 dps, and therefore with deficiency $= 0$; in the case $D = 1$, it is a quartic with 2 dps, and therefore deficiency $= 2$; in the case $D = 2$ it is a quintic with a triple point $= 3$, and a double point $= 1$, together 4 dps, and therefore deficiency $= 2$. Hence in every case the new curve has the same deficiency as the original curve.

12. The theorem thus is, that the given curve of the order n, with deficiency D, may be rationally transformed into a curve of an order depending only on the deficiency, and having the same deficiency with the given curve, viz.: $D=0$, the new curve is of the order $2 (=D+2)$; $D=1$, it is of the order $4 (=D+3)$; $D=2$, it is of the order $5 (=D+3)$; and $D>2$, it is for D odd, of the order $D+3$; and for D even, of the order $D+2$. It will presently appear that these are not the lowest values which it is possible to give to the order of the new curve. Riemann's object was, not that the order of the transformed curve might be as low as possible, but that the equation in (ξ, η) might be in each of these parameters separately of the lowest possible order; and this he effected by giving to the transformed curve the two μ-tuple points.

13. It is to be noticed that the theorem, that for any rational transformation of one curve into another the two curves have the same deficiency, is in effect given (as a consequence of Riemann's theory) by Clebsch in the Paper, "Ueber die Singularitäten algebraischer Curven." —Crelle, t. 64, pp. 98—100. I have, by the assistance of a formula communicated to me by Dr. Salmon, obtained a direct analytical demonstration of this theorem.

14. I remark that (x, y, z) being connected by an equation, if $(x : y : z)$ are given rationally in terms of $(\xi : \eta : \zeta)$, then it follows that $(\xi : \eta : \zeta)$ are also expressible rationally in terms of $(x : y : z)$. It is convenient to consider the transformation from this point of view, and I now proceed to the independent development of the theory, as follows:—

15. We have a given curve $U = (x, y, z)^n = 0$, with deficiency D,

which is by the transformation $\xi : \eta : \zeta = P : Q : R$ (where P, Q, R are given functions $(x, y, z)^k$ each of the same order k) transformed into the curve $\Upsilon = (\xi, \eta, \zeta)^\nu = 0$. The transformed curve has, as we know, the same deficiency D as the original curve.

16. To find the order of the tranformed curve, we must find the number of its intersections with an arbitrary line $a\xi + b\eta + c\zeta = 0$. Writing in this equation $\xi : \eta : \zeta = P : Q : R$, we obtain the equation $aP + bQ + cR = 0$, and combining therewith the equation $U = 0$, the two equations, being of the orders k and n respectively, give kn systems of values of $(x : y : z)$; and to each of these, in virtue of the equations $\xi : \eta : \zeta = P : Q : R$, there corresponds a single set of values of $(\xi : \eta : \zeta)$, and therefore a single point of intersection; hence the number of intersections, that is, the order of the transformed curve, is $= kn$.

17. If, however, the curves $P = 0$, $Q = 0$, $R = 0$ meet in an ordinary point of the curve $U = 0$, then it is easy to see there is a reduction $= 1$ in the foregoing value; and so, if they meet in a dp of the curve $U = 0$, then there is a reduction $= 2$. More generally, if the curves $P = 0$, $Q = 0$, $R = 0$ each pass through the same α dps and β ordinary points of the curve $U = 0$, then there is a reduction $= 2\alpha + \beta$. In fact, the curve $aP + bQ + cR = 0$ meets the curve $U = 0$ in kn points; but among these are included the α dps, each counting as 2 intersections, and the β points; the number of the remaining intersections is $= kn - 2\alpha - \beta$, and the order of the transformed curve is equal to this number.

I assume that we have $k < n$:

18. A curve of the order k may be made to pass through $\frac{1}{2}k(k+3)$ points; it is moreover known that, if any three curves, $P = 0$, $Q = 0$, $R = 0$, of the order k each pass through the same $\frac{1}{2}k(k+3) - 1$ points, then the three curves have all their intersections common, the equations being, in fact, connected by an identical relation of the form $\alpha P + \beta Q + \gamma R = 0$. To make the order of the transformed curve as low as possible, we must make the curves $P = 0$, $Q = 0$, $R = 0$ meet on the curve $U = 0$ in as many points as possible, and it appears from the remark just made, that the greatest possible number is $= \frac{1}{2}k(k+3) - 2$; in particular, for $k = n-1$, $n-2$, $n-3$, the number of points on the curve $U = 0$ will be at most equal to $\frac{1}{2}(n^2+n) - 3$, $\frac{1}{2}(n^2-n) - 3$, $\frac{1}{2}(n^2-3n) - 2$, respectively.

19. Hence, considering the curve $U = 0$ with deficiency D, or with $\frac{1}{2}(n^2-3n) - D + 1$ dps, *first*, if $k = n-1$, we may assume that the transforming curves $P = 0$, $Q = 0$, $R = 0$ of the order $n-1$, each pass

through the	$\frac{1}{2}(n^2-3n) - D + 1$	dps
and through	$2n + D - 4$	other points
together,	$\frac{1}{2}(n^2+n) - 3$	points of the curve $U = 0$.

This being so, each of the three curves will meet the curve $U = 0$.

in the dps counting as	$n^2-3n-2D+2$	points,
in the	$2n+ D-4$	points,
and in	$D+2$	other points,
together	n^2-n	points;

whence the order of the transformed curve is $= D+2$.

20. In precisely the same manner, *secondly*, if $k = n-2$, then we may assume that the transforming curves $P = 0$, $Q = 0$, $R = 0$, of the order $n-2$, each pass

through the	$\frac{1}{2}(n^2-3n)-D+1$	dps,
and through	$n +D-4$	other points,
together	$\frac{1}{2}(n^2-n) \quad -3$	points of the curve $U = 0$.

And, this being so, each of the three curves will meet the curve $U = 0$

in the dps counting as	$n^2-3n-2D+2$	points,
in the	$n+ D-4$	points,
and in	$D+2$	other points,
together	n^2-2n	points;

whence the order of the transformed curve is also in this case $=D+2$.

21. I was under the impression that the order of the transformed curve could not be reduced below $D+2$, but it was remarked to me by Dr. Clebsch, that in the case $D>2$, the order might be reduced to $D+1$. In fact, considering, *thirdly*, the case $k = n-3$, we see that the transforming curves $P=0$, $Q=0$, $R=0$ of the order $n-3$ may be made to pass

through he	$\frac{1}{2}(n^2-3n)-D+1$	dps,
and through	$D-3$	other points,
together	$\frac{1}{2}(n^2-3n) \quad -2$	points of the curve $U = 0$;

and, this being so, each of the three curves meets the curve $U = 0$

in the dps counting as	$(n^2-3n)-2D+2$	points,
in the	$D-3$	points,
and in	$D+1$	other points,
together	2^2-3n	points;

whence the order of the transformed curve is in this case $= D+1$.

22. The general theorem thus is, that a curve of the order n with deficiency D can be, by a transformation of the order $n-1$ or $n-2$, transformed into a curve of the order $D+2$; and if $D>2$, then the

given curve can be, by a transformation of the order $n-3$, transformed to a curve of the order $D+1$; the transformed curve having in each case the same deficiency D as the original curve.

23. In particular, if $D = 1$, a curve of the order n with deficiency 1, or with $\frac{1}{2}(n^2-3n)$ dps, can be transformed into a cubic curve with the same deficiency, that is with 0 dps, or the given curve can be transformed into a cubic. This case is discussed by Clebsch in the Memoir "Ueber diejenige Curven deren Coordinaten elliptische Functionen eines Parameters sind"—Crelle, t. 64, pp. 210, 271. And he has there given in relation to it a theorem which I establish as follows :—

24. Using the transformation of the order $n-1$, if besides the $2n+D-4$ $(=2n-3)$ points on the given curve $U = 0$, we consider another point O on the curve, then we may, through the $\frac{1}{2}(n^2-3n)$ dps, the $2n-3$ points, and the point O, draw a series of curves of the order $n-1$; viz., if P_0, Q_0, R_0, are what the functions P, Q, R become on substituting therein for (x, y, z) the coordinates (x_0, y_0, z_0) of the given point O, then the equation of any such curve will be $aP+bQ+cR = 0$, with the relation $aP_0+bQ_0+cR_0$ between the parameters a, b, c; or (what is the same thing), eliminating c, the equation will be $a(PR_0-P_0R)+b(QR_0-Q_0R) = 0$, which contains the single arbitrary parameter $a : b$. In the cubic which is the transformation of the given curve, we have a point O′ corresponding to O, and if (ξ_0, η_0, ζ_0) be the coordinates of this point, then, corresponding to the series of curves of the order $n-1$, we have a series of lines through the point O′ of the cubic, viz., the lines $a\xi+b\eta+c\zeta = 0$, with the relation $a\xi_0+b\eta_0+c\zeta_0 = 0$ between the parameters; or, what is the same thing, we have the series of lines $a(\xi\zeta_0-\zeta\xi_0)+b(\eta\zeta_0-\zeta\eta_0) = 0$, containing the same single parameter $a : b$. By determining this parameter, the curves of the order $n-1$ will be the curves of this order through the dps, the $2n-3$ points, and the point O, which touch the given curve $U = 0$; and the lines will be the tangents to the cubic from the point O′. As the number of tangents to a cubic from a point on the cubic is $= 4$, it is clear that the values of the parameters $a : b$ will be determined by a certain quartic equation; and there will of course be 4 tangent curves of the order $n-1$ corresponding to the 4 tangents to the cubic. Now, by Dr. Salmon's anharmonic property of the tangents of a cubic, if on the cubic we vary the position of the point O′, the absolute invariant $I^3 \div J^2$ of the quartic in $(a : b)$ remains unaltered; that is, the absolute invariant $I^3 \div J^2$ of the quartic which determines the 4 tangent curves of the order $n-1$, is independent of the position of the point O on the given curve $U=0$; and since the tangent curves in question have the same relation to each of the $2n-3$ points and to the points O, it follows that the invariant is also independent of the position of each of the $2n-3$ points; that is, we have the following theorem, viz. :—

25. Considering a curve of the order n with deficiency $= 1$; we may through the $\frac{1}{2}(n^2-3n)$ dps, and through any $2n-2$ points on the curve, draw, so as to touch the curve, four curves of the order $n-1$; viz., these are given by an equation $aP'+bQ'=0$, where the ratio $a:b$ is determined by a certain quartic equation $(*\,\mathfrak{Q}a, b)^4 = 0$; then *theorem*, the absolute invariant $I^3 \div J^2$ of the quartic function is independent of the positions of the $2n-2$ points on the curve $U = 0$, and it is consequently a function of only the coefficients of the curve $U = 0$, being, as is obvious, an absolute invariant of the curve $U = 0$.

26. And moreover, if the curve $U = 0$ is, by a transformation of the order $n-1$, by means of $2n-3$ points on the curve as above, transformed into a cubic, then the absolute invariant $I^3 \div J^2$ of the quartic equation, which determines the tangents to the cubic from any point O′ on the cubic (or, what is the same thing, the absolute invariant $S^3 \div T^2$ of the cubic, taken with a proper numerical multiplier), is independent of the positions of the $2n-3$ points on the curve $U = 0$, being in fact equal to the above-mentioned absolute invariant of the curve $U = 0$. The like results apply to the transformation of the order $n-2$.

27. Suppose now that we have $D > 2$, and consider a curve of the order n with the deficiency D, that is, with $\frac{1}{2}(n^2-3n)-D+1$ dps, transformed by a transformation of the order $n-3$ into a curve of the order $D+1$ with deficiency D; then, assuming the truth of the subsidiary theorem to be presently mentioned, it may be shown, by very similar reasoning to that above employed, that the absolute invariants of the transformed curve of the order $D+1$ (the number of which is $=4D-6$) will be independent of the positions of the $D-3$ points used in the transformation, and will be equal to absolute invariants* of the given curve $U = 0$.

28. The subsidiary theorem is as follows: Consider a curve of the order $D+1$, with deficiency D, that is, with $\frac{1}{2}D(D-1)-D = \frac{1}{2}(D^2-3D)$ dps; the number of tangents to the curve from any point O′ on the curve is $= (D+1)D-(D^2-3D)-2$, $= 4D-2$, (this assumes, however, that the dps are proper dps, not *cusps*,) the pencil of tangents has $4D-5$ absolute invariants, and of these all but one, that is $4D-6$, absolute invariants of the pencil are independent of the position of the point O′ on the curve, and are respectively equal to absolute invariants of the curve.

29. To establish it, I observe that a curve of the order $D+1$ with deficiency D, or with $\frac{1}{2}(D^2-3D)$ dps, contains $\frac{1}{2}(D+1)(D+4)-\frac{1}{2}(D^2-3D)$, $= 4D+2$ arbitrary constants, and it may therefore be made to satisfy $4D+2$ conditions. Now imagine a given pencil of

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* It is right to notice that the absolute invariants spoken of here and in what follows are not in general rational ones.

4D−2 lines, and let a curve of the form in question be determined so as to pass through the centre of the pencil, and touch each of the 4D−2 lines; the curve thus satisfies 4D−1 conditions, and its equation will contain $4D+2-(4D-1), = 3$ arbitrary constants. But if we have any particular curve satisfying the 4D−1 conditions, then by transforming the whole figure homologously, taking the centre of the pencil as pole and any arbitrary line as axis of homology, so as to leave the pencil of lines unaltered (analytically, if at the centre of the pencil $x = 0$, $y = 0$, then by writing $\alpha x+\beta y+\gamma z$ in place of z), the transformed curve still satisfies the 4D−1 conditions, and we have, by the homologous transformation introduced into its equation, 3 arbitrary constants; that is, we have obtained the most general curve which satisfies the conditions in question. The absolute invariants of the general curve are independent of the 3 arbitrary constants introduced by the homologous transformation; and they are consequently functions of only the co-efficients of the given pencil of 4D−2 lines; this being so, it is obvious that they will be respectively equal to absolute invariants of the pencil of 4D−2 lines. The number of the absolute invariants of the general curve of the order D+1 is $= \frac{1}{2}(D+1)(D+4)+1-9$; but there is a reduction = 1, for each of the dps, hence in the present case the number is $\frac{1}{2}(D+1)(D+4)-\frac{1}{2}(D^2-3D)-8, = 4D-6$; and there are thus 4D−6 absolute invariants of the curve, each of them equal to an absolute invariant of the pencil; that is, of the 4D−5 absolute invariants of the pencil, there are 4D−6, each of them equal to an absolute invariant of the curve, and consequently independent of the position of the point O′ on the curve; which is the theorem which was to be proved. I believe the reasoning is quite correct, but there are some points in it which require further examination; it is therefore given subject to any correction which may hereafter appear to be necessary.

30. The general subject may be illustrated by considerations belonging to solid geometry. If we imagine the original curve and the transformed curve as situate in different planes, then, joining each point of the original curve with the corresponding point on the transformed curve, we have a series of lines forming a scroll (skew surface): if the two curves are of the orders n, n' respectively, then the complete section by the plane of the original curve is made up of this curve of the order n, and of n' generating lines; and similarly the complete section by the plane of the transformed curve is made up of this curve of the order n', and of n generating lines. Conversely, given a scroll of the order $n+n'$, any two sections of this scroll, being in general curves of the same order $n+n'$, are rational transformations the one of the other; but for the *general* scroll of the order $n+n'$, it is not possible to find sections breaking up as above.

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ON

DIFFERENTIAL RESOLVENTS.*

BY THE REV. ROBERT HARLEY, F.R.S., F.R.A.S.,

PROFESSOR OF MATHEMATICS AND LOGIC IN AIREDALE COLLEGE, BRADFORD.

1. My previously published papers relating to the class of linear differential equations, which are now known by the name "Differential Resolvents," will be found in the following works :—

A. *Manchester Proceedings.* (Session 1861–62.) Vol. II. pp. 181—184, 199—203, 237—241. "Remarks on the Theory of Equations."

B. *Quarterly Journal of Mathematics.* (1862.) Vol. V. pp. 337—360. "On the Theory of the Transcendental Solution of Algebraic Equations."

C. *Manchester Memoirs.* (Session 1861–1862.) Series III. Vol. II. pp. 232—245. "On a Certain Class of Linear Differential Equations."

I shall refer to these papers by the letters prefixed.

2. By studying some of my results, the late Professor Boole was led to perceive their relation to a certain general theorem, which it will be convenient here to announce.

Boole's Theorem.—If u represent the mth power of any root of the algebraic equation

$$y^n - xy^{n-1} - 1 = 0,$$

then u, considered as a function of x, satisfies the linear differential equation

$$n\left[x\frac{d}{dx}\right]^n u + \left[\frac{n-1}{n}x\frac{d}{dx} + \frac{m}{n} - 1\right]^{n-1}\left(x\frac{d}{dx} - m - n\right)x^n u = 0,$$

in which the ordinary factorial notation

$$[a]^b = a\,(a-1)\,(a-2)\ldots(a-b+1)$$

is adopted.

And the complete integral of the above differential equation will be

$$u = C_1 y_1^m + C_2 y_2^m \ldots + C_n y_n^m,$$

* Read Nov. 20, 1865.

$y_1, y_2 \ldots y_n$ being the n roots of the given algebraic equation. See

D. *Philosophical Transactions* for 1864, pp. 733–755. "On the Differential Equations which determine the form of the Roots of Algebraic Equations."*

3. I have lately found that the process of demonstration employed by Professor Boole may be extended so as to establish the following more general theorem.

Theorem I.—The differential equation

$$a^r [\mathrm{D}]^n u - [\mathrm{E}]^{n-r} [\mathrm{F}]^r x^n u = 0,$$

in which $\mathrm{D} = x\frac{d}{dx}$, $\mathrm{E} = \frac{n-r}{n}\mathrm{D} + \frac{m}{n} - 1$, and $\mathrm{F} = \frac{r}{n}\mathrm{D} - \frac{m}{n} - 1$,

is satisfied by the mth power of any root of the algebraic equation

$$y^n - xy^{n-r} + a = 0,$$

u being considered as a function of x.

I shall hereafter show that this theorem implicitly involves the following:

Theorem II.—The differential equation

$$n^n [\mathrm{G}]^{n-r} [\mathrm{D}]^r u - (n-1)^r [\mathrm{H}]^n x^r u = 0,$$

in which $\mathrm{G} = \frac{n-r}{r}\mathrm{D} - \frac{m}{r}$ and $\mathrm{H} = \frac{n}{r}\mathrm{D} - \frac{m}{r} - 1$,

is satisfied by the mth power of any root of the algebraic equation

$$y^n - ny^{n-r} + (n-1)\,x = 0.$$

4. I proceed to establish these theorems. It will be convenient to to begin with the form

$$y^n - xy^{n-r} - a = 0,$$

for which, if we write $y^n = z$, we have

$$u = z^{\frac{m}{n}} = (a + xz^{\frac{n-r}{n}})^{\frac{m}{n}};$$

and, developing by Lagrange's theorem,

$$u = u_0 + u_1 x + u_2 x^2 + \&c., \textit{ ad infinitum},$$

where, in general,

$$u_t = \frac{1}{[t]^t} \cdot \frac{d^{t-1}}{da^{t-1}} \left(a^{\frac{t(n-r)}{n}} \frac{d}{da} a^{\frac{m}{n}} \right)$$

$$= \frac{1}{[t]^t} \cdot \frac{m}{n} \cdot \left[\frac{m + n(t-1) - rt}{n} \right]^{t-1} a^{\frac{m-rt}{n}};$$

* See also Boole's "Differential Equations," Supplementary volume (a work which has been published since this Paper was read), pp. 190-199.

so that

$$u_{t-n} = \frac{1}{[t-n]^{t-n}} \cdot \frac{m}{n} \left[\frac{m+n(t-1)-rt}{n} - (n-r) \right]^{t-n-1} a^{\frac{m-rt}{n}+r}.$$

Whence, observing that in general

$$[\phi]^{t-1} = [\phi]^{n-r} [\phi-(n-r)]^{t-n-1} [\phi+r-t+1]^r,*$$

we have

$$a^r [t]^n u_t - \left[\frac{n-r}{n} t + \frac{m}{n} - 1 \right]^{n-r} \left[-\frac{r}{n} t + \frac{m}{n} + r \right]^r u_{t-n} = 0,$$

which, since

$$\left[-\frac{r}{n} t + \frac{m}{n} + r \right]^r = (-)^r \left[\frac{r}{n} t - \frac{m}{n} - 1 \right]^r,$$

reduces to

$$a^r [t]^n u_t - (-)^r \left[\frac{n-r}{n} t + \frac{m}{n} - 1 \right]^{n-r} \left[\frac{r}{n} t - \frac{m}{n} - 1 \right]^r u_{t-n} = 0.$$

This determines the law of the co-efficients in the Lagrangean development, and gives us the differential equation

$$a^r \left[x \frac{d}{dx} \right]^n u - (-)^r \left[\frac{n-r}{n} x \frac{d}{dx} + \frac{m}{n} - 1 \right]^{n-r} \left[\frac{r}{n} x \frac{d}{dx} - \frac{m}{n} - 1 \right]^r x^n u = 0,$$

which, when $-a$ is written for a, coincides with the first form exhibited in the last article, and establishes Theorem I.

It will be observed that Professor Boole's equation (Art. 2) is a particular case of the above: in fact, if in the above we write $r = 1$, it reduces to Professor Boole's equation, as it ought to do.

5. Next, to establish Theorem II. The equation

$$y^n - xy^{n-r} - a = 0$$

will be changed into

$$y_1^n - n y_1^{n-r} + (n-1) x_1 = 0,$$

if we make

$$y = \left\{ -\frac{a}{n-1} \right\}^{\frac{1}{n}} \frac{y_1}{x_1^{\frac{1}{n}}}, \qquad x = n \left\{ -\frac{a}{n-1} \right\}^{\frac{r}{n}} \cdot \frac{1}{x_1^{\frac{r}{n}}}$$

and, for the transformation of the above differential equation, we have

$$u = \left\{ -\frac{a}{n-1} \right\}^{\frac{m}{n}} \frac{u_1}{x_1^{\frac{m}{n}}}$$

in which $u_1 = y_1^m$. These substitutions being effected, we arrive, after

* In dealing with these factorial forms, it is useful to remember that

$$[a]^{b+c} = [a]^b [a-b]^c.$$

some reductions, at the equation

$$n^n \left[\frac{n-r}{r} x_1 \frac{d}{dx_1} - \frac{m}{r}\right]^{n-r} \left[x_1 \frac{d}{dx_1}\right]^r u_1 - \left[\frac{n}{r} x_1 \frac{d}{dx_1} - \frac{m}{r} - 1\right]^n x_1^r u_1 = 0,$$

which establishes Theorem II. (Art. 3).

6. By the aid of Theorem II., I have been enabled to verify a great variety of results which I had previously obtained by induction. Some of these results I have already published (see A, B, and C); and others, which will be noticed further on, I hope to be in a condition to publish, together with the details of calculation, before long. The value of the differential co-efficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., expressed as rational and integral functions of the dependent variable y, possess an interest and importance altogether apart from the peculiar form of the linear differential equations to which they conduct us. I may mention here that the calculation of some of these co-efficients, especially for the higher equations, involves a great amount of labour—much greater, in fact, than the simplicity of the final results may seem to indicate.

7. When $m = 1$, we have $u = y$, and the differential equations in Art. 3 become

$$a^r [\mathrm{D}]^n y - [\mathrm{E}]^{n-r} [\mathrm{F}]^r x^n y = 0,$$

and

$$n^n [\mathrm{G}]^{n-r} [\mathrm{D}]^r y - (n-1)^r [\mathrm{H}]^n x^r y = 0 \quad \text{(I.)},$$

in which D remains unchanged $\left(= x \frac{d}{dx}\right)$, and

$$\mathrm{E} = \frac{n-r}{n} \mathrm{D} + \frac{1}{n} - 1,$$

$$\mathrm{F} = \frac{r}{n} \mathrm{D} - \frac{1}{n} - 1,$$

$$\mathrm{G} = \frac{n-r}{r} \mathrm{D} - \frac{1}{r},$$

$$\mathrm{H} = \frac{n}{r} \mathrm{D} - \frac{1}{r} - 1.$$

The first of these differential equations is satisfied by any one of the roots of the algebraic equation

$$y^n - xy^{n-r} + a = 0,$$

and the second by any one of the roots of

$$y^n - ny^{n-r} + (n-1)x = 0 \quad \text{(1)},$$

y being considered in both as a function of x.

8. Now the theory of differential resolvents suggests (see B, p. 339)

that, for all positive and integral values of n and r, n being greater than r, the above differential equations must admit of being immediately depressed at least one order. This, in fact, will be found to be the case.

9. In order to save space, we shall, throughout the remainder of this paper, deal exclusively with the form (I.), Art. 7; the other form at the head of that article may, of course, be treated in precisely the same way, and with corresponding results.

10. If we make $r = 1$, it is easy to see that the symbolical functions $[G]^{n-r}$ and $[H]^n$ will contain a common factor, viz. $D-1$; for in that case

$$[G]^{n-r} = [(n-1)\,D-1]^{n-1},$$

the last factor in the expansion of which is $(n-1)\,(D-1)$, and

$$[H]^n = [nD-2]^n,$$

the last factor but one in which is $n\,(D-1)$. Therefore, when $r = 1$, the form (I.) becomes, by immediate integration,

$$n^{n-1}\,[(n-1)\,D]^{n-1}\,y - (n-1)\,(nD-n-1)\,[nD-2]^{n-2}\,xy = (D-1)^{-1}\,0 = Cx.$$

Perhaps the readiest way of determining the constant C is the following:—If we write $z+1$ in place of y, the algebraic equation

$$y^n - ny^{n-1} + (n-1)\,x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a)$$

[which is what (1) becomes when we substitute in it unity for r] will be transformed into another equation wanting in its second term; and therefore the differential resolvent of this transformed equation

$$n^{n-1}[(n-1)\,D]^{n-1}(z+1) - (n-1)\,(nD-n-1)[nD-2]^{n-2}\,x\,(z+1) = Cx,$$

which reduces to

$$n^{n-1}[(n-1)\,D]^{n-1}z - (n-1)\,(nD-n-1)\,[nD-2]^{n-2}\,xz + [n-1]^{n-1}x = Cx,$$

can contain no term independent of z. (See C, p. 245.) We have, consequently,

$$C = [n-1]^{n-1},$$

and the differential resolvent of (a) is completely determined. The result agrees with one which I arrived at some time ago by induction. (See C, p. 239 *et seq.*)

11. Similarly, by making $r = n-1$, and applying Theorem II., we may with equal facility confirm the following result, which I originally obtained by induction (see A and B), and which Mr. Cayley after-

wards established rigorously by means of Lagrange's theorem (see C, pp. 111—113);* viz., that the differential resolvent of

$$y^n - ny + (n-1)\,x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

is $$n^{n-1}\,[\mathrm{D}]^{n-1}y - (n-1)^{n-1}\left[\frac{n}{n-1}\,\mathrm{D} - \frac{2n-1}{n-1}\right]^{n-1} x^{n-1}y = 0,$$

for all values of n other than 2. In the case of $n=2$, the resolvent contains a term independent of y; this term written on the right-hand side is equal to x, and the terms on the left-hand follow the law above indicated.

12. If we take $r = 2$, the algebraic equation (1) becomes

$$y^n - ny^{n-2} + (n-1)\,x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

and the differential equation (I.), Art. 7, reduces to

$$n^n\left[\frac{n-2}{2}\,\mathrm{D} - \frac{1}{2}\right]^{n-2}[\mathrm{D}]^2 y - (n-1)^2\left[\frac{n}{2}\,\mathrm{D} - \frac{3}{2}\right]^n x^2 y = 0 \ \ldots\ldots\ (c'),$$

which, when n is odd, admits of an immediate first integral; for, in this case, the symbolical factor $(\mathrm{D}-1)$ is common to both terms in the first member, and we have

$$2n^{n-1}\mathrm{D}\left[\frac{n-2}{2}\,\mathrm{D} - \frac{1}{2}\right]^{n-2} y - (n-1)^2\left[\frac{n}{2}\mathrm{D} - \frac{3}{2}\right]^{\frac{n-3}{2}}\left[\frac{n}{2}\mathrm{D} - \frac{n}{2} - 1\right]^{\frac{n+1}{2}} x^2 y = 0.$$

This confirms a particular result which I arrived at, by direct calculation, some months ago, but which I have not hitherto published; viz., that the differential resolvent of the trinomial quintic equation

$$y^5 - 5y^3 + 4x = 0$$

is $$5^4\,\mathrm{D}\left[\frac{3}{2}\mathrm{D} - \frac{1}{2}\right]^3 y - 2^3\left(\frac{5}{2}\mathrm{D} - \frac{3}{2}\right)\left[\frac{5}{2}\mathrm{D} - \frac{7}{2}\right]^3 x^2 y = 0.$$

When n is even, the differential equation (c') does not appear to admit of immediate integration, but it can be shown that, in this case, it may be reduced ultimately from the nth to the $\frac{n}{2}$th order. I was led to suspect that it might be thus reduced by the following considerations. I found by direct calculation that the differential resolvent of the trinomial quartic equation

$$y^4 - 4y^2 + 3x = 0$$

may be reduced to

$$4\,[2\mathrm{D}]^2 y - 3\left[2\mathrm{D} - \frac{3}{2}\right]^2 xy = 0,$$

---

* See also D, p. 738, where Professor Boole deduces the result referred to in the text from his theorem.

and that of the trinomial sextic equation

$$y^6 - 6y^4 + 5x = 0$$

may be reduced to

$$6^3 \, \mathrm{D}\left[2\mathrm{D} - \frac{1}{2}\right]^2 y - 5\left[3\mathrm{D} - \frac{3}{2}\right]^3 xy = 0.$$

And these results suggested to me that perhaps it might be possible to reduce (c'), when n is even, to the form

$$n^{\frac{1}{2}n} \, \mathrm{D}\left[\frac{n-2}{2}\mathrm{D} - \frac{1}{2}\right]^{\frac{n-2}{2}} y - (n-1)\left[\frac{n}{2}\mathrm{D} - \frac{3}{2}\right]^{\frac{1}{2}n} xy = 0 \ldots\ldots\ldots\ (c'').$$

13. Professor Cayley has since shown me that this reduction is possible. In a letter to me, bearing date 28th September, 1865, and written a few days after I sent him my results, he says:—

"Writing $2n$ for n, your two forms of the differential equation for

$$y^{2n} - 2ny^{2n-2} + (2n-1)\,x = 0$$

are $(\mathrm{D} = x\delta_x)$

$$(2n)^n\left[(n-1)\,\mathrm{D} - \frac{1}{2}\right]^{n-1} \mathrm{D}y - (2n-1)\left[n\mathrm{D} - \frac{3}{2}\right]^n xy = 0 \ldots\ldots\ldots(1),$$

and $(2n)^{2n}\left[(n-1)\mathrm{D} - \frac{1}{2}\right]^{2n-2} \mathrm{D}\,(\mathrm{D}-1)\,y - (2n-1)^2\left[n\mathrm{D} - \frac{3}{2}\right]^{2n} x^2y = 0$

$\ldots\ldots\ldots(2).$

"Representing the first of these by

$$\mathrm{F}\,(\mathrm{D})\,y - \mathrm{G}\,(\mathrm{D})\,xy = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then the second is

$$\mathrm{F}\,(\mathrm{D})\,\mathrm{F}\,(\mathrm{D}-1)\,y - \mathrm{G}\,(\mathrm{D})\,\mathrm{G}\,(\mathrm{D}-1)\,x^2y = 0 \ldots\ldots\ldots\ (2).$$

Now, in general, $\mathrm{F}\,(\mathrm{D}-1)\,xy = x\mathrm{F}\,(\mathrm{D})\,y,$

$$\mathrm{G}\,(\mathrm{D}-1)\,x^2y = x\mathrm{G}\,(\mathrm{D})\,xy,$$

and, therefore, if $\mathrm{F}\,(\mathrm{D})\,y = \mathrm{G}\,(\mathrm{D})\,xy \ldots\ldots\ldots\ldots\ldots\ldots(1),$

then $\mathrm{F}\,(\mathrm{D}-1)\,xy = \mathrm{G}\,(\mathrm{D}-1)\,x^2y,$

that is, $\mathrm{G}\,(\mathrm{D})\,\mathrm{F}\,(\mathrm{D}-1)\,xy = \mathrm{G}\,(\mathrm{D})\,\mathrm{G}\,(\mathrm{D}-1)\,x^2y,$

or $\mathrm{F}\,(\mathrm{D})\,\mathrm{F}\,(\mathrm{D}-1)\,y = \mathrm{G}\,(\mathrm{D})\,\mathrm{G}\,(\mathrm{D}-1)\,x^2y \ldots\ldots\ldots(2),$

which proves the identity of the two equations."

14. The differential equation for

$$y^n - ay^{n-r} + bx = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where a and b are any constants, is

$$a^n\,[\mathrm{G}]^{n-r}\,[\mathrm{D}]^r\,y - b^r\,[\mathrm{H}]^n\,x^ry = 0 \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{II}).$$

(See Art. 7.) For the algebraic equation (1) is changed into (2) by writing $\left(\frac{n}{a}\right)^{\frac{n}{r}} \cdot \frac{bx}{n-1}$ and $\left(\frac{n}{a}\right)^{\frac{1}{r}} y$ for x and y respectively; and these substitutions being made in the differential equation (I.) the symbols D, G, H remaining for these substitutions unchanged, we pass immediately to the form (II.)

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ON

RADIAL CURVES.*

BY R. TUCKER, M.A.

V.

1. My previous article upon *Radial Curves* will be found in the "Mathematical Reprints from the *Educational Times*," Vol. I., pp. 16—19; Vol. II., pp. 27—38; and Vol. IV., pp. 22—28.

Occasional reference is made to these (under the titles *A*, *B*, *C*, respectively) in the following remarks.

2. It will be sufficient for my present purpose to consider the radial as generated in the following manner:—

Let P and Q be corresponding points on a curve and its evolute; then, O being the origin, if the parallelogram of which OP, PQ are adjacent sides be completed, the locus of the remaining angular point (R) will be the radial.

Hence, given the curves on which two of the three points lie, the locus of the third angular point will be the remaining curve.

3. The area between a curve, its evolute, and the corresponding radii of curvature, is easily shown to be

$$\tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta.$$

4. It may readily be seen that my method can be applied to the determination of properties of circles of curvature which are orthogonal, thus, by Art. 18 (*A*), it is seen that, in the case of the *cycloid*, the distance between the centres of two such circles is constant and equal to twice the diameter of the generating circle of the primitive curve. Hence also we can obtain properties of such circles in the case of the conic sections.

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* Read Dec. 18, 1865. Professor De Morgan in the Chair.

5. If we take a curve such that its abscissa and ordinate are equal respectively to the arc of the primitive curve and its evolute, and denote the corresponding arc of the new curve by S, we have, by equation (μ), (B)—

$$\left(\frac{d\mathrm{S}}{ds}\right)^2 = 1+\left(\frac{ds'}{ds}\right)^2 = \left(\frac{ds''}{ds}\right)^2,$$

and, therefore, $d\mathrm{S} = ds''$. Hence it appears that differences of arc of this curve equal corresponding differences of arc of the radial.

6. Taking the well-known formula (Todhunter, *Diff. Calc.*, § 323), and employing the notation of Arts. 60, 63 (B), we have, if R′ stand for the radius of curvature of the evolute of the evolute,

$$\rho_1 = \frac{(\rho^2+\mathrm{R}^2)^{\frac{3}{2}}}{\rho^2+2\mathrm{R}^2\mp\rho\mathrm{R}'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\phi).$$

This result may be put under another form,

since $$\rho_1\frac{d\phi_1}{d\phi} = \sqrt{\rho^2+\mathrm{R}^2} \ldots\ldots\ldots\ldots\ldots\ (\xi) \text{ in } (B),$$

$$\therefore \quad \frac{d\phi_1}{d\phi} = 1+\frac{\mathrm{R}^2\mp\rho\mathrm{R}'}{\rho^2+\mathrm{R}^2} = \frac{\rho^2+2\mathrm{R}^2\mp\rho\mathrm{R}'}{\rho^2+\mathrm{R}^2} \ldots\ldots\ldots\ldots\ldots\ldots(\phi').$$

7. From equation (τ) (C), we get

$$2\frac{d\phi}{d\phi_1} = \frac{\rho\mathrm{R}_1+\mathrm{R}\rho_1}{\rho_1\mathrm{R}}.$$

Combining this with (φ'), there results

$$2\rho_1\mathrm{R}\,(\rho^2+\mathrm{R}^2) = (\rho^2+2\mathrm{R}^2\mp\rho\mathrm{R}')\,(\rho\mathrm{R}_1+\mathrm{R}\rho_1),$$

or $$\rho_1\mathrm{R}\,(\rho\pm\mathrm{R}') = \mathrm{R}_1\,(\rho^2+2\mathrm{R}^2\mp\rho\mathrm{R}')\ldots\ldots\ldots\ldots\ldots\ldots(\psi).$$

8. In this and following Articles it is proposed to find the radials for the general equations of the second and third degrees.

The most general form of the equation of the second degree is

$$a_1x^2+2b_2xy+c_3y^2+2d_1x+2e_1y+f_1 = 0 \quad \ldots\ldots\ldots\ldots\ (\mathrm{i.})$$

Differentiating twice, and writing p, q for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, there results

$$\mathrm{A}_1x+\mathrm{B}_1y+\mathrm{C}_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{ii.}),$$

$$\mathrm{A}_2x+\mathrm{B}_2y+\mathrm{C}_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{iii.}),$$

where $\mathrm{A}_1 = b_2p+a_1$, $\mathrm{B}_1 = c_3p+b_2$, $\mathrm{C}_1 = e_1p+d_1$,

$\mathrm{A}_2 = b_2q$, $\mathrm{B}_2 = c_3q$, $\mathrm{C}_2 = 2b_2p+c_3p^2+e_1q+a_1$;

from (iii.), $$y = \lambda x+\mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (\mathrm{iv.}),$$

if $$\lambda = -\frac{\mathrm{A}_2}{\mathrm{B}_2}, \quad \mu = -\frac{\mathrm{C}_2}{\mathrm{B}_2}.$$

Substituting (iv.) in (i.), the result is

$$Ax^2+Bx+C=0 \quad \text{.....(v.)},$$

if $A=a_1+2b_2\lambda+c_3\lambda^2$, $B=2b_2\mu+2c_3\lambda\mu+2d_1+2e_1\lambda$,

$$C=c_3\mu^2+2e_1\mu+f_1;$$

and (ii.) similarly becomes $A'x+C'=0$ .....(vi.),

where $A'=A_1+B_1\lambda$, $C'=B_1\mu+C_1$.

Eliminating x from (v.) and (vi.), the resulting equation is

$$AC'^2-A'BC'+A'^2C=0,$$

or $(3b_2c_3p^2+2b_2^2p+c_3^2p^3+b_2e_1q+a_1c_3p+a_1b_2-c_3d_1q)^2$

$+2q\,(c_3d_1-b_2e_1)\,(3b_2c_3p^2+2b_2^2p+c_3^2p^3+b_2e_1q+a_1c_3p+a_1b_2-c_3d_1q)$

$+(a_1c_3-b_2^2)\{c_3^2p^4+4b_2c_3p^3+(4b_2^2+2a_1c_3)p^2+4a_1b_2p+a_1^2$

$+(f_1c_3-e_1^2)q^2\}=0.$

This reduces to

$$c_3^3p^6+6b_2c_3^2p^5+3c_3(a_1c_3+4b_2^2)p^4+4b_2(3a_1c_3+2b_2^2)p^3$$

$$+a_1(3a_1c_3+8b_2^2)p^2+6a_1^2b_2p$$

$$+a_1^3+\{f_1(a_1c_3-b_2^2)-a_1e_1^2-c_3d_1^2+2b_2d_1e_1\}q^2=0.$$

9. Hence the radial in this case is a curve of the sixth degree, and may be found from the above by writing $\frac{x}{y}$ for p, and $\frac{x^2+y^2}{y^3}$ for q.

If the given curve be a parabola, or $a_1c_3=b_2^2$, the resulting equation, by (8), is

$$c_3^2p^3+3b_2c_3p^2+3b_2^2p+a_1b_2=\pm q\,(c_3d_1-b_2e_1).$$

10. Taking now the most general form of the equation of the third degree

$$a_1x^3+b_2y^3+3a_2x^2y+3b_1xy^2+3a_3x^2+6dxy+3b_3y^2+3c_1x+3c_2y+c_3=0$$

.....(i.),

and proceeding as before, the resulting equations are

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0 \quad \text{.....(ii.)},$$

$$A'y^2+B'xy+C'x^2+D'y+E'x+F'=0 \quad \text{.....(iii.)},$$

where $\begin{cases} A=3b_2p+3b_1, & B=6b_1p+6a_2, & C=3a_2p+3a_1, \\ D=6b_3p+6d, & E=6dp+6a_3, & F=3c_2p+3c_1, \end{cases}$

and $\begin{cases} A'=3b_2q, & B'=6b_1q, & C'=3a_2q, \\ D'=2Ap+B+6b_3q, & E'=Bp+2C+6dq, & F'=Dp+E+3c_2q; \end{cases}$

from (ii.) and (iii.) there results

$$(A'B)\,xy+(A'C)\,x^2+(A'D)\,y+(A'E)\,x+(A'F)=0,$$

$$\therefore \qquad -y = \frac{\lambda x^2+\mu x+\nu}{\mu' x+\nu'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

if $$\left.\begin{array}{l} \lambda = (A'C), \quad \mu = (A'E), \quad \nu = (A'F) \\ \mu' = (A'B), \quad \nu' = (A'D) \end{array}\right\}.$$

Substitute (α) in (i.), and the resulting equation is

$$A''x^6+B''x^5+C''x^4+D''x^3+E''x^2+F''x+G'' = 0 \ldots\ldots\ldots (iv.),$$

where

$$\begin{aligned}
A'' &= a_1\mu'^3-\lambda^3 b_2-3a_2\mu'^2\lambda+3b_1\lambda^2\mu', \\
B'' &= 3a_1\mu'^2\nu'-3b_2\lambda^2\mu-3a_2(\mu'^2\mu+2\lambda\mu'\nu')+3a_3\mu'^3 \\
&\quad +3b_1(\lambda^2\nu'+2\lambda\mu\mu')-6d\lambda\mu'^2+3b_3\lambda^2\mu', \\
C'' &= 3a_1\mu'\nu'^2-3b_2(\lambda^2\nu+\mu^2\lambda)-3a_2(\mu'^2\nu+2\mu\mu'\nu'+\lambda\nu'^2) \\
&\quad +3b_1(\mu^2\mu'+2\lambda\mu\nu'+2\lambda\mu'\nu)+9a_3\mu'^2\nu'+3c_1\mu'^3-3c_2\lambda\mu'^2 \\
&\quad -6d(\mu\mu'^2+2\lambda\mu'\nu')+3b_3(\lambda^2\nu'+2\lambda\mu\mu'), \\
D'' &= a_1\nu'^3-b_2(\mu^3+6\lambda\mu\nu)-3a_2(\mu\nu'^2+2\mu'\nu\nu')+9a_3\mu'\nu'^2 \\
&\quad +3b_1(\mu^2\nu'+2\lambda\nu\nu'+2\mu\mu'\nu)+3b_3(2\lambda\mu\nu'+\mu^2\mu'+2\lambda\mu'\nu) \\
&\quad -6d(\lambda\nu'^2+2\mu\mu'\nu'+\mu'^2\nu)+9c_1\mu'^2\nu'+c_3\mu'^3 \\
&\quad -3c_2(\mu\mu'^2+2\lambda\mu'\nu'), \\
E'' &= -3b_2(\lambda\nu^2+\mu^2\nu)-3a_2\nu\nu'^2+3b_1(\mu'\nu^2+2\mu\nu\nu')+3a_3\nu'^3 \\
&\quad -6d(\mu\nu'^2+2\mu'\nu\nu')+9c_1\mu'\nu'^2+9c_3\mu'^2\nu' \\
&\quad +3b_3(\mu^2\nu'+2\lambda\nu\nu'+2\mu\mu'\nu)-3c_2(\lambda\nu'^2+2\mu\mu'\nu'+\mu'^2\nu), \\
F'' &= -3b_2\mu\nu^2+3b_1\nu^2\nu'-6d\nu\nu'^2+3c_1\nu'^3+3c_3\mu'\nu'^2 \\
&\quad +3b_3(\mu'\nu^2+2\mu\nu\nu')-3c_2(\mu\nu'^2+2\mu'\nu\nu'), \\
G'' &= -b_2\nu^3+3b_3\nu^2\nu'-3c_2\nu\nu'^2+c_3\nu'^3.
\end{aligned}$$

Also, the result of substituting (α) in (ii.), (iii.) is

$$A_1x^4+B_1x^3+C_1x^2+D_1x+E_1 = 0 \ldots\ldots\ldots\ldots\ldots (v.),$$

$$a'x^4+b'x^3+c'x^2+d'x+e' = 0 \ldots\ldots\ldots\ldots\ldots (vi.);$$

if
$$\begin{aligned}
A_1 &= A\lambda^2-B\lambda\mu'+C\mu'^2, \\
B_1 &= 2A\lambda\mu-B(\lambda\nu'+\mu\mu')+2C\mu'\nu'-D\lambda\mu'+E\mu'^2, \\
C_1 &= A(\mu^2+2\lambda\nu)-B(\mu\nu'+\mu'\nu)+C\nu'^2-D(\lambda\nu'+\mu\mu')+2E\mu'\nu'+F\mu'^2, \\
D_1 &= 2A\mu\nu-B\nu\nu'-D(\mu\nu'+\mu'\nu)+E\nu'^2+2F\mu'\nu', \\
E_1 &= A\nu^2-D\nu\nu'+F\nu'^2,
\end{aligned}$$

Similarly, a', b', &c. are immediately obtained from the above by changing A into A', &c.

Now, multiplying (v.) successively by x^2 and x, and eliminating from (iv.) and the resulting equation, we arrive at the equation

$$ax^4+bx^3+cx^2+dx+e = 0 \ldots\ldots\ldots\ldots\ldots (vii.),$$

where
$$\left.\begin{aligned} a &= A_1(A_1C'') - B_1(A_1B'') \\ b &= A_1(A_1D'') - C_1(A_1B'') \\ c &= A_1(A_1E'') - D_1(A_1B'') \\ d &= A_1^2 F'' \quad - E_1(A_1B'') \\ e &= A_1^2 G'' \end{aligned}\right\};$$

whence, eliminating between (vi.) and (vii.), we get the known result

$$\left.\begin{array}{llll} (ab'), & (ac'), & (ad'), & (ae') \\ (ac'), & (ad')+(bc'), & (ae')+(bd') & (be') \\ (ad'), & (ae')+(bd'), & +(be')+(cd'), & (ce') \\ (ae'), & (be'), & (ce'), & (de'). \end{array}\right\};$$

which is therefore the equation to the radial required.

11. The degree of the radial in this case will be found not to exceed 4×32, or the 128th. I was induced to examine the degrees of the radials in this Article and Article 9, in consequence of a suggestion to this effect kindly made by Professor Cayley.

12. It is readily seen how to proceed to find the radials for higher curves.

GENERAL MEETING

OF

THE LONDON MATHEMATICAL SOCIETY,

HELD AT UNIVERSITY COLLEGE, ON JAN. 15, 1866.

(PROFESSOR DE MORGAN IN THE CHAIR.)

Mr. MORGAN JENKINS (in the absence of the Secretaries) read a report, the substance of which was as follows:—

The Society consisted at its formation of twenty-seven members, and held its first meeting in University College on January 16, 1865, Professor De Morgan being in the Chair. The majority, indeed nearly all of these, were members of University College; and the present members of the Society would doubtless feel obliged to the original members of the Society for the part they took in its establishment.

The Society had lost two of its members by death, namely—

Mr. Benj. Gompertz, F.R.S., F.R.A.S., an eminent mathematician, and especially well known for the discovery of an important formula with respect to the law of mortality.

Mr. H. J. Purkiss, Principal of the Royal School of Naval Architecture, Kensington, a mathematician of great promise, whose career had been cut short by his being drowned whilst bathing at Cambridge.

The number of members (including those who had just been elected) was sixty-nine.

The Papers which had been read to the Society during the year were—

On Determinants	By Mr. Ranyard.
Strictures on the Laws of Motion... ...	„ Mr. Bompas.
On Geometrical Inversion...	„ Professor Hirst.
Mr. Gray's Method of forming Logarithms	„ Mr. Adler.

Mr. Byrne's System of Dual Arithmetic...	By Mr. Grimley.
On the regular Hypocycloidal Tricusp ...	„ Mr. Jenkins.
An elementary proof and generalization of Newton's hitherto undemonstrated Rule for the Discovery of Imaginary Roots...	„ Professor Sylvester.
On Transformation of Plane Curves ...	„ Professor Cayley.
Sketch of a Chapter in Modern Geometry	„ Mr. Greer.
On Radial Curves	„ Mr. Tucker.
On Motion in a Circle, and its relation to Planetary Motion	„ Professor Sylvester.

The state of the finances of the Society was shown by the following account:—

Receipts.	£.	s.	d.	Expenditure.	£.	s.	d.
Subscriptions for Session ending June, 1865..... ..	16	10	0	Printing 3 Papers read to the Society £8 10 0			
Subscriptions for Session ending June, 1866.....	20	15	0	Notices, Cards, &c... 4 6 9			
					12	16	9
				Attendance at Meetings, postage, &c.....	3	3	6
				Balance in hand	21	4	9
Total	£37	5	0	Total.....	£37	5	0

	£.	s.	d.
Balance in hand	21	4	9
Subscriptions due	14	10	0
Total	£35	14	9

The recommendation of the Committee, that a room should be obtained, if possible, in Burlington House, was then discussed.

It was finally proposed by Professor Sylvester, and seconded by Mr. Roberts, and carried—

That steps be taken to ascertain, on behalf of the Society, whether, and on what terms, rooms can be obtained at Burlington House.

It was proposed by Mr. Jenkins, and seconded by Mr. Thos. Cotterill, and carried—

That Members shall have the option of paying a single life composition of Ten pounds in place of the annual subscription of Ten shillings.

It was proposed by Professor Sylvester, and seconded by Professor Hirst, and carried—

That the Committee be empowered to spend a sum not exceeding ten pounds upon mathematical journals.

February 19, 1866.

A Meeting held at University College (Professor de Morgan in the Chair.)

The following gentlemen were duly proposed and seconded for election at the next meeting as members of the Society:—

Messrs. Fowler, Lambert, Merrifield, Shackleton, and Taylor.

The following gentlemen were duly elected members of the Society:—

Messrs. J. Ashton, A. Cohen, C. F. Coutts, E. A. Hadley, R. Hunter, B. Smith, J. M. Solomon, and J. Stirling.

The following Paper was then read by Mr. Cotterill; after which Professor De Morgan communicated a proof of the theorem, previously discussed at Meetings of the Society, that every function has a root.

ON

CERTAIN PROPERTIES OF PLANE POLYGONS OF AN EVEN NUMBER OF SIDES.

By T. COTTERILL, M.A.,

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

The object of the following paper is to prove certain common properties of the areas of polygons of $2n$ sides, by means of their simplest form, the quadrilateral, with a view hereafter to show that the expressions obtained in the cases of the quadrilateral and hexagon can be generalised into forms which, without any change of symbols, admit of dual interpretations, both on the plane and on the sphere.

1. The area of a plane quadrilateral ABCD is invariable, so long as its diagonals AC, BD and the angle between them remain the same, and this is easily seen either geometrically or more simply by its trigonometrical expression, one half the product of the diagonals and the sine of its included angle. The area vanishes if the extremities of a diagonal coincide, or if the two diagonals are parallel.

Hence its area can be denoted by $\begin{pmatrix} AC \\ BD \end{pmatrix}$; and if P be any point in its plane, we shall have the following equations from the geometrical definition of the area,

$$\begin{aligned} \begin{pmatrix} AC \\ BD \end{pmatrix} &= PAB+PBC+PCD+PDA \\ &= ABC+ACD = BCD+BDA \\ &= ABD-CBD = DAC-BAC, \end{aligned}$$

in which it is important to remark the connexion between the diagonal representation of the area and that of its component triangle, especially with regard to the rule of signs.

Thus $$\begin{pmatrix} AC \\ BD \end{pmatrix} = -\begin{pmatrix} CA \\ BD \end{pmatrix} = -\begin{pmatrix} BD \\ AC \end{pmatrix}, \text{ \&c.}$$

If one diagonal remains fixed whilst the other is constant in magnitude

but variable in position, the area varies in magnitude and sign with the sine of an angle. Hence the sum of the squares of the areas of two quadrilaterals whose diagonal angles differ by a right angle is constant, and if one vanishes the other is a maximum.

2. The area Γ of the hexagon AFBDCE, in which ABC and DEF are the vertices of the alternate triangles formed from it, is given by the equation

$$\Gamma = \mathrm{PAF}+\mathrm{PFB}+\mathrm{PBD}+\mathrm{PDC}+\mathrm{PCE}+\mathrm{PEA},$$

which can be expressed in two ways as the sum of three quadrilaterals by means of equations such as

$$\mathrm{PAF}+\mathrm{PFB} = \begin{pmatrix}\mathrm{PF}\\ \mathrm{AB}\end{pmatrix} \text{ and } \mathrm{PEA}+\mathrm{PAF} = \begin{pmatrix}\mathrm{PA}\\ \mathrm{EF}\end{pmatrix}, \text{ \&c.}$$

$$\text{Hence } \Gamma = \begin{pmatrix}\mathrm{PF}\\ \mathrm{AB}\end{pmatrix}+\begin{pmatrix}\mathrm{PD}\\ \mathrm{BC}\end{pmatrix}+\begin{pmatrix}\mathrm{PE}\\ \mathrm{CA}\end{pmatrix} = \begin{pmatrix}\mathrm{PA}\\ \mathrm{EF}\end{pmatrix}+\begin{pmatrix}\mathrm{PB}\\ \mathrm{FD}\end{pmatrix}+\begin{pmatrix}\mathrm{PC}\\ \mathrm{DE}\end{pmatrix}$$

$$= \mathrm{ABC}-\mathrm{DBC}-\mathrm{ECA}-\mathrm{FAB} = \mathrm{DEF}-\mathrm{AEF}-\mathrm{BFD}-\mathrm{CDE}$$

$$= \begin{pmatrix}\mathrm{EF}\\ \mathrm{AB}\end{pmatrix}-\begin{pmatrix}\mathrm{DE}\\ \mathrm{BC}\end{pmatrix} = \begin{pmatrix}\mathrm{FD}\\ \mathrm{BC}\end{pmatrix}-\begin{pmatrix}\mathrm{EF}\\ \mathrm{CA}\end{pmatrix} = \begin{pmatrix}\mathrm{DE}\\ \mathrm{CA}\end{pmatrix}-\begin{pmatrix}\mathrm{FD}\\ \mathrm{AB}\end{pmatrix}.$$

As any two of the quadrilaterals in which P is a point can be made to disappear, it follows that, in the case of the vanishing hexagon, the lines drawn through the vertices of each of its alternate triangles parallel to the sides of the other are concurrent, and the directions of the sides are therefore in Involution.

When this is not the case, let lines be drawn through the vertices of one of the triangles, as ABC, parallel to the sides of the other DEF. Thus, let CXY be parallel to DE; AYZ to EF; and BZX to FD.

Then $\mathrm{CDE} = \mathrm{XDE} = \mathrm{YDE} = c$, $\mathrm{AEF} = \mathrm{YEF} = \mathrm{ZEF} = a$,

$$\mathrm{BFD} = \mathrm{ZFD} = \mathrm{XFD} = b, \text{ suppose;}$$

$$\text{and } \Gamma = \begin{pmatrix}\mathrm{XA}\\ \mathrm{EF}\end{pmatrix}+\begin{pmatrix}\mathrm{XB}\\ \mathrm{FD}\end{pmatrix}+\begin{pmatrix}\mathrm{XC}\\ \mathrm{DE}\end{pmatrix} = \begin{pmatrix}\mathrm{XA}\\ \mathrm{EF}\end{pmatrix} = \mathrm{XEF}-\mathrm{AEF}$$

$$= \mathrm{XEF}-a = \mathrm{YFD}-b = \mathrm{ZDE}-c.$$

Therefore $\mathrm{XEF} = \Gamma+a$, $\mathrm{YFD} = \Gamma+b$, $\mathrm{ZDE} = \Gamma+c$.

The area of the triangle XYZ is found from the areal co-ordinates of its points referred to DEZ by the following equation:

$$(\mathrm{DEF})^2(\mathrm{XYZ}) = \begin{Bmatrix}\Gamma+a, & b, & c\\ a, & \Gamma+b, & c\\ a, & b, & \Gamma+c\end{Bmatrix}.$$

The Determinant is evaluated at once, from observing that the three rows coincide if Γ vanish, and that the sum of the constituents of a row

is $\Gamma+a+b+c$, or (DEF). Hence, observing the factors in the principal diagonal, we have

$$(\mathrm{DEF})^2(\mathrm{XYZ}) = \Gamma^2(\mathrm{DEF}) \quad \text{or} \quad \Gamma^2 = (\mathrm{DEF})(\mathrm{XYZ}).$$

Similarly, if the sides of the triangle (TUV) circumscribe the triangle (DEF), and are respectively parallel to the sides of (ABC),

$$\Gamma^2 = (\mathrm{ABC})(\mathrm{TUV}) = (\mathrm{DEF})(\mathrm{XYZ}).$$

Hence, if through the vertices of each of two triangles in a plane lines are drawn respectively parallel to the sides of the other, the products of the similar triangles are the same and equal to the square of the hexagon, formed from them in the manner above described. Hence also, if a triangle is fixed, the produce of the areas of the inscribed and circumscribing similar and similarly situated triangles is constant.

In order to trace the changes in the areas of the hexagon corresponding to those of the quadrilateral from varying the position of its diagonals, suppose, first, the triangle ABC to be fixed, whilst DEF is moved in its own plane by a motion of translation. As the component quadrilaterals remain constant in magnitude, the area of a hexagon is unaltered by a motion of translation of its alternate triangles. If Γ_θ denotes its area when the angle between corresponding sides of the triangle is increased by θ, $\Gamma_{\frac{1}{2}\pi} = \frac{1}{2}\Sigma\mathrm{PA},\ \mathrm{EF}\cos(\mathrm{PA},\ \mathrm{EF}) = \frac{1}{2}\Sigma\mathrm{PD},\ \mathrm{BC}\cos(\mathrm{PD},\ \mathrm{BC})$, and $\Gamma_\theta = \Gamma\cos\theta = \Gamma_{\frac{1}{2}\pi}\sin\theta$, and the vanishing position is given by $\tan\theta = \Gamma \div \Gamma_{\frac{1}{2}\pi}$. The properties, therefore, of a quadrilateral dependent on the variation of its diagonal angle have counterparts in the hexagon, the alternate triangles taking the place of the diagonals in a peculiar manner.

There is another curious analogy between the two figures AB, CD,

$$\cos(\mathrm{AB},\ \mathrm{CD}) = \tfrac{1}{2}(\mathrm{AD}^2+\mathrm{BC}^2-\mathrm{AC}^2-\mathrm{BD}^2).$$

Hence, in a maximum quadrilateral the sums of the squares of the opposite sides are equal, and if it vanish, the difference of the sum of the squares of its opposite sides is greater than in any other relative position of the diagonals. $\Gamma_{\frac{1}{2}\pi}$ is the sum of three such terms, and if they are expressed in the same way, the P's will disappear, and $\Gamma_{\frac{1}{2}\pi}$ is found equal to one-fourth the difference of the sums of the squares of the alternate sides of the original hexagon. Hence, if a hexagon be a maximum, the sums of the squares of its alternate sides are equal, and the difference of these sums is the greatest in the vanishing hexagon.

Other properties can be deduced from these expressions, arising chiefly from the arbitrary position of the point P, but as they, for the most part, depend on the goniometrical function of Involution, they scarcely come within the scope of this Paper.

3. From a polygon of $2n$ sides, two alternate polygons of n sides each can be formed, and its area expressed in two ways as the sum of n quadrilaterals, each containing the same arbitrary point. Thus, all the properties which have been proved in the hexagon (except those which depend on the alternate polygons being triangles) are true for polygons of an even number of sides. The general expression, of which $\Sigma PA, EF \cos(PA, EF) = \Sigma PD, BC \cos(PD, BC)$ is a type, is also $= \Sigma DA, EF \cos(DA, EF) = \Sigma AD, BC \cos(AD, BC)$, and is remarkable as being true for general space.

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*A Proof that every Function has a Root.**

By Professor DE MORGAN, F.R.A.S.

That *every* function has a root is a proposition which is never either affirmed or denied; but it is assumed in the universal practice of taking inversion for granted; that is, in allowing $x = \phi^{-1}y$ to be a consequence of $y = \phi x$, whatever may be the value of x.

The proof now described is easily given, when divested of all elucidation of difficulty, and of all guard against objection. If $\phi(x+y\sqrt{-1}) = \mathrm{P}+\mathrm{Q}\sqrt{-1}$, P and Q being real and continuous, then, the values of x and y being known to be definite values of P and Q, the changes of x and y due to $d\mathrm{P}$ and $d\mathrm{Q}$ can be found from the second pair in

$$d\mathrm{P} = \mathrm{P}_x dx + \mathrm{P}_y dy, \qquad \mathrm{R}dx = \mathrm{Q}_y d\mathrm{P} - \mathrm{P}_y d\mathrm{Q},$$
$$d\mathrm{Q} = \mathrm{Q}_x dx + \mathrm{Q}_y dy, \qquad \mathrm{R}dy = -\mathrm{Q}_x d\mathrm{P} + \mathrm{P}_x d\mathrm{Q},$$

where P_x means $d\mathrm{P} : dx$, and $\mathrm{R} = \mathrm{P}_x\mathrm{Q}_y - \mathrm{P}_y\mathrm{Q}_x$. If, then, $x = \lambda$, $y = \mu$ give $\mathrm{P} = l$, $\mathrm{Q} = m$, and if from the point (l, m) to the point (a, b) we draw a curve $\mathrm{Q} = f\mathrm{P}$, and, beginning at $\mathrm{P} = l$, $\mathrm{Q} = m$, and $x = \lambda$, $y = \mu$, we determine successive values of dx, dy, answering to successive values of $d\mathrm{P}$, $d\mathrm{Q}$, we may construct by *polygonal approximation* a curve, the second end of which shows $x = \alpha$, $y = \beta$, the values which give $\mathrm{P} = a$, $\mathrm{Q} = b$, or which solve the equation

$$\phi(x+y\sqrt{-1}) = a+b\sqrt{-1},$$

provided always that the calculation be nowhere interrupted by any failure of the second pair of equations.

Now the functions P, Q are so related that $\mathrm{P}_x = \mathrm{Q}_y$, $\mathrm{P}_y = -\mathrm{Q}_x$, $\mathrm{R} = \mathrm{P}_x^2 + \mathrm{Q}_y^2$. The second pair of equations can fail only when P_x and P_y both vanish, and when one or both become infinite. It is easily shown that one cannot become infinite without the other, except for isolated pairs of values of x and y. And, as to the rest, it is as easily shown that the families of curves $\mathrm{P}_x = h$, $\mathrm{P}_y = k$ are orthogonal, that no one ever meets another except at right angles; that is, that there can never be any curve common to the two equations, nor

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* Read January 15, 1866.

anything but isolated points. Consequently, the curve $Q = fP$ being taken so as to avoid these points of difficulty, the solution of

$$\phi(x+y\sqrt{-1}) = a+b\sqrt{-1}$$

may be established, as above, and even approximately obtained. Observe that ϕz need not be real when z is real: that is, it need not be true that $\phi(x-y\sqrt{-1}) = P-Q\sqrt{-1}$.

March 19, 1866.

Ordinary Meeting held at University College (Professor De Morgan in the Chair).

The following gentlemen were proposed and seconded for election as members at the next meeting of the Society:—

Dr. M. Friedlander, Dr. Salmon, and Rev. R. Townsend.

The following gentlemen were elected members of the Society:—

Messrs. Fowler, Lambert, Merrifield, Shackleton, and Taylor.

The Papers read were:—

"On various properties of Confocal Cartesian Ovals," by Mr. Crofton.

"On the Centres of Algebraical Curves and Surfaces," by Mr. Roberts.

Professor Sylvester gave a short account of a theorem on the rotation of a body under the action of no forces, of which the following is an abstract.

*On an Addition to Poinsot's Ellipsoidal Mode of representing the Motion of a Rigid Body turning freely round a Fixed Point, whereby the Time may be made to register itself mechanically.**

By Professor SYLVESTER, F.R.S.

The author of this communication alluded to the well-known geometrical representation invented by Poinsot for exhibiting the motion of a rigid body acted on by no forces, through the medium of an ellipsoid, whose principal moments of inertia are equal to those of the rotating body, and which is supposed to roll without swinging upon a fixed plane, its centre remaining fixed. He pointed out, as an imperfection in this mode of representation, that whilst it exhibits the geometrical path of the body, it gives no image to the mind of the time in which any portion of such path is performed; the velocity of rotation in any position, it is true, is in a manner represented proportionally by the length of the radius vector drawn from the fixed centre to the point of contact of the ellipsoid with the fixed plane; but of the time there is no indication afforded other than what can be inferred analytically from this law of velocity by means of an integration. The author explained how this imperfection could be remedied, and the time put distinctly in evidence, and conceived as registering itself on a dial plate as the ellipsoid continues to roll. For this purpose, the part of the ellipsoid remote from the fixed plane may be conceived to be pared away until its form becomes that of a portion of an ellipsoid confocal with the surface in contact with the fixed plane. This confocal part of the surface may be conceived to be brought into contact with a fixed plate parallel to the fixed plane, and capable of only one motion, viz., of revolution round an axis, which, if produced, would pass through the fixed centre, and perpendicular to the fixed plane. It was explained how the ellipsoid in the act of rolling upon the last-named plane would roll upon the plate parallel to it, and at the same time by the friction drive the plate round its axis: this angular rotation could be measured upon a fixed dial-face immediately above the plate, and the amount of it would be always *precisely proportional* to the time that would be occupied by the

* Read March 19, 1866. Professor De Morgan in the Chair.

body which the ellipsoid represents, if perfectly free, in passing from any given position to any other; that is to say, the representative ellipsoid passing from a position A to a position B, and in the act of passing from A to B, driving the rotating plate through the angle U, U would measure the time in which the free body could pass from A to B. If a^2, b^2, c^2 be the semi-axes squared of Poinsot's ellipsoid, $a^2-\lambda$, $b^2-\lambda$, $c^2-\lambda$ of the confocal one above described, L the initial impulsive moment of the free body, the time between the position A and the position B will be measured by $\frac{U}{L\lambda}$; so that, the initial impulse being supposed given, the relation between the time and the magnitude of the divisions in the dial-plate is invariable; and thus, by supposing the fixed plane and the parallel plate to admit, by a preliminary adjustment, of being set at any required distance from one another, the same instrument would continue to measure on the same scale the time of the free body passing from any initial position whatever to any other position into which it could turn.

The shortness of time (subsequent to the reading of the two other papers of the evening) remaining over for this exposition precluded the author of it from entering into further developments of the subject.

*On Certain Properties of the Cartesian Ovals, treated by the Method of Vectorial Coordinates.**

By M. W. CROFTON, B A.,
MATHEMATICAL MASTER IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

1. The Cartesian ovals seem at all times to have attracted considerable attention, though no great success has attended the attempts made to investigate their properties. Of these, the most remarkable which has been discovered is, no doubt, that of the third focus, given by Chasles without proof, which evidently teaches us that, as there is another focus possessing the same properties and the same importance as the two comprised in the definition, we should extend our attention to all three if we wish for a general and complete view of the curve. The unsuitable nature of the coordinates, polar or rectangular, which have been employed, is doubtless the reason of the limited progress which has been made in the investigation of these curves. Vectorial coordinates, the distances of a point from two fixed foci or poles, seem to be naturally indicated by their very definition. The method has difficulties of its own, and, indeed, has yet to be created; but every step made in the study of these coordinates will necessarily be a step in the theory of the Cartesian ovals, whose equation, being linear, fills the same place in this system which that of the straight line does in rectangular coordinates.

It will be useful to point out here one especial difficulty and inconvenience which rectangular coordinates present (and polar in an equal degree), viz., that we cannot represent the curve $\rho + m\rho' = 2a$ without extraneous factors. Thus, if $a^2y^2+b^2x^2 = a^2b^2$ represent an ellipse, and if (xy) be any point in its plane, whose vectorial coordinates, referred to the foci, are (ρ, ρ'), it may be easily proved that

$$a^2y^2+b^2x^2-a^2b^2 = \tfrac{1}{16}\left\{(\rho+\rho')^2-4a^2\right\}\left\{4a^2-(\rho-\rho')^2\right\}.$$

Hence, $a^2y^2+b^2x^2 = a^2b^2$ represents the *four* curves $\pm\rho \pm\rho' = 2a$, of which three are imaginary. For the hyperbola, two are real, viz., $\rho-\rho' = 2a$, $\rho'-\rho = 2a$; so that, as is worth remarking, in the vectorial system the two branches of the hyperbola are distinct curves. Now the rectangular equation of the Cartesian oval $\rho+m\rho' = 2a$

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* Read March 19, 1866. Professor De Morgan in the Chair.

will be of the fourth degree, and will represent the four curves $\pm\rho \pm m\rho' = 2a$. Two of these, in general, are real, being *conjugate* ovals, one within the other; and great inconvenience arises from both being mixed up in one equation. M. Quetelet has been led into an error, as we shall see, by the polar equation, which offers the same difficulty.

The present paper contains a few properties of these curves, which have been arrived at by the method of which I speak.

2. I begin by a very simple proof of Chasles's property; first stating the following well-known theorem—

If A, B, C be three points in directum, P any other point,

$$AP^2.BC + CP^2.AB = BP^2.AC + AB.AC.BC \quad \ldots\ldots\ldots (1).$$

Now let P be any point on a Cartesian oval whose equation, referred to ff' as foci ($ff' = 2c$), is

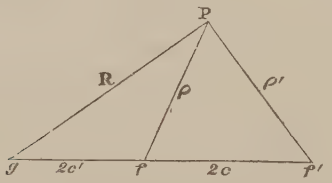
$$\rho + m\rho' = 2a \quad \ldots\ldots\ldots (2);$$

take any point (g) in ff', let $Pg = R$, $fg = 2c'$; we have (1)

$$2cR^2 + 2c'\rho'^2 = 2(c+c')\rho^2 + 8cc'(c+c');$$

eliminate ρ' by (2),

$$cR^2 = (c+c')\rho^2 - \frac{c'}{m^2}(2a-\rho)^2 + 4cc'(c+c').$$



This is the vectorial equation of the curve, referred to g, f as poles; R, ρ being the vectors. If we arrange the second member according to powers of ρ, we can determine c' so as to render it a perfect square: the value will be

$$c' = \frac{a^2 - m^2c^2}{c(m^2-1)} * \quad \ldots\ldots\ldots (3);$$

then, taking the square root, the equation will become linear in R, ρ; by a little reduction, and using the value (3) for c', it may be thus written:—

$$\pm mR = \frac{a}{c}\rho + 2c' \quad \ldots\ldots\ldots (4).$$

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* Hence $\rho + m\rho' = 2a$ will represent a family of triconfocal Cartesians, if the constants m, a satisfy the condition

$$\frac{m^2c^2 - a^2}{m^2-1} = \text{constant}.$$

Which sign we ought to take in the different cases requires discussion; but I pass this over for the present. We see then that if $fg\ (=2c')$ has the value in (3), g will be a third focus: the equation of the curve referred to it, and either f or f', being linear.

From (3) we find the condition that the biconfocal ovals

$$\rho + m\rho' = 2a,$$

$$\rho + \mu\rho' = 2a,$$

shall have their third focus common, to be

$$\mu^2(a^2-c^2) - m^2(a^2-c^2) = a^2-a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

As in (3) the *squares* only of m and a enter, it follows that the quadruple system of ovals,

$$\pm\rho \pm m\rho' = 2a,$$

two of which are generally real, are triconfocal.

If $m = \pm 1$, we have, from (3), $c' = \infty$; hence, in the conics, the third focus is infinitely distant. It is easily shown that the linear relation between the vectors to this focus and either of the others, reduces in this case to the familiar property of the distances from the focus and directrix.

3. The general equation of a straight line in vectorial coordinates is complicated. Hence, to find the equation of the tangent to a curve seems to be a problem of little interest or utility in this system. But there is a general class of *circles*, viz., all those whose centres lie on the *axis* joining the poles or foci of reference, whose equation is extremely simple. For want of a better name, we may call them *axial* circles. Let C be any point on ff'; let $fC = m$, $f'C = n$, and draw a circle with centre C and any radius $CP = l$; let $fP = \rho$, $f'P = \rho'$, and, from (1), the equation of the circle will be

$$\left.\begin{array}{l} n\rho^2 + m\rho'^2 = 2c(l^2+mn) \\ \text{or} \quad \rho^2 + k\rho'^2 = A \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

k, A being constants. Hence any linear equation in ρ^2, ρ'^2 represents an axial circle. If $k = -1$, we have $\rho^2 - \rho'^2 = A$, the general equation of a straight line perpendicular to the axis.

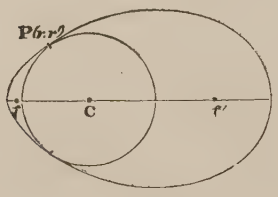
Now we may easily find the *axial circle touching the Cartesian* $\rho + m\rho' = 2a$ at a given point (r, r'). As it passes through this point its equation is of the form

$$\rho^2 + k\rho'^2 = r^2 + kr'^2;$$

determining k by the further condition that $\frac{d\rho}{d\rho'}$ is the same at P for

both curves, and, reducing, we shall find the required equation to be

$$\frac{\rho^2}{r} + m\frac{\rho'^2}{r'} = 2a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7).$$



As in our present system, everything is necessarily similar above and below the axis, every point being accompanied by its *image* on the other side, this circle has *double* contact with the oval, as in the figure.

The equation (7) has many important applications in the present theory, and we shall have occasion to return to it further on. Its resemblance to the equation of the curve is striking. I proceed now to apply it in proving the following theorem :—

4. *Any system of triconfocal Cartesian ovals intersect each other orthogonally.* Such a system (as we shall see more fully further on), consists of two series of curves, one having the extreme right-hand focus external to it, and the other two internal; the other having the left-hand focus external. No two curves belonging to the same series can meet; but every curve meets all those of the other series—orthogonally, as we shall show—the analogy to confocal conics being obvious.

Let $$\rho + m\rho' = 2a, \qquad \rho + \mu\rho' = 2a$$

be any two biconfocal ovals, intersecting in the point (r, r'). Their tangent axial circles there are

$$\frac{\rho^2}{r} + m\frac{\rho'^2}{r'} = 2a, \qquad \frac{\rho^2}{r} + \mu\frac{\rho'^2}{r'} = 2a.$$

The condition that these circles shall cut orthogonally is*

$$4c^2(m+\mu) = 2a(r'+\mu r) + 2a(r'+mr);$$

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* From Equation (6), we may easily find the radii, and distance of centres, of the two circles, $\rho^2 + k\rho'^2 = A, \quad \rho^2 + k'\rho'^2 = A';$
and thence show that the condition that they shall meet at right angles is

$$4c^2(k+k') = A(1+k') + A'(1+k).$$

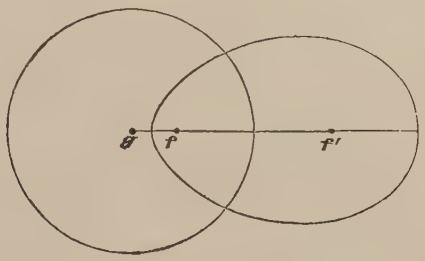
A property of frequent utility is that, if T be the tangent from $(\rho\rho')$ to the circle $\rho^2 + k\rho'^2 = A$, $\qquad T^2 = \frac{\rho^2 + k\rho'^2 - A}{1+k}.$

eliminating r, r' by means of the relations $r+mr'=2a$, $r+\mu r'=2a$,

we find $$\mu^2(a^2-c^2)-m^2(a^2-c^2)=a^2-\alpha^2;$$

showing (5) that the ovals have their third focus common.

5. A circle is really a Cartesian oval, having, for three foci, its centre and any two inverse points; for, referring it to the inverse points, we have $\rho=k\rho'$; and, again, to one of them and the centre,



we have $0.\rho+R=l$, the radius. Hence, if g, f, f' are the foci of a system of triconfocal ovals, the circle from g as centre, having f', f as inverse points, is one of the curves of the system; so that the family of ovals (meaning, of course, the series which meets it) are cut orthogonally by this circle; or, as we may state the theorem—

From an extreme focus (g) tangents are drawn to a system of triconfocal Cartesians: all these tangents are equal.

Of course there is a second circle, whose centre is f', and in which f, g are inverse points, which also belongs to the same family of ovals, but to the opposite series, cutting orthogonally the former circle, and all those ovals which have f' as external focus.

These two circles are of great importance in the present theory. We shall call them the *confocal circles* with reference to any oval.

6. To find the relation between the segments $f'P$, $f'p$ of any transversal drawn from the external focus f' to cut the curve.

Let the equation of the oval, referred to f, f', be

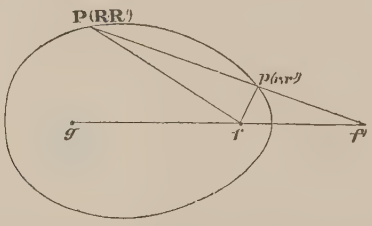
$$\rho+m\rho'=2a;$$

let the coordinates of P be (R, R'); of p, (r, r'); then we have

$$\left.\begin{aligned} &R+mR'=2a, \quad r+mr'=2a; \\ \text{also, by (1),} \quad &R^2r'+4c^2(R'-r')=r^2R'+R'r'(R'-r'); \end{aligned}\right\}$$

eliminating R, r, and remembering the value of c' in (3), we easily find

$$R'r' = 4c(c+c').$$

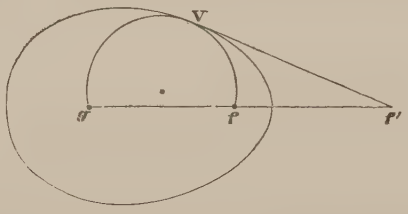


Hence, $f'1 \,.fp = f'g \,.f'f$; so that the rectangle of the segments of any transversal from the external focus is constant; and not only for the particular oval, but for all triconfocal ones. We have then the theorem—

Every Cartesian oval is its own inverse with regard to the confocal circle whose centre is its external focus. Hence, many theorems would follow from the doctrine of inverse curves.*

7. From Art. 6 it is plain that, *if any circle be drawn through the two internal foci g, f, cutting the curve in two points* P, *p on the same side of the axis*, P*p produced passes through the external focus f'*.

If a circle be drawn through g, f, *touching* the curve in V, the tangent at V passes through f'. This point V is also that where the curve is cut by the orthotomic confocal circle; it is also, as we shall see, the point of maximum or minimum curvature, and seems to correspond



to the extremity of the minor axis in the ellipse. Thus it is clearly a very remarkable point on the curve; for convenience' sake we may call it the *apex*, reserving *vertex* for the points where the curve cuts the axis.

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* In the *Philosophical Magazine* for January, 1866, the following remarkable theorem in Dynamics is given by Professor Sylvester:—

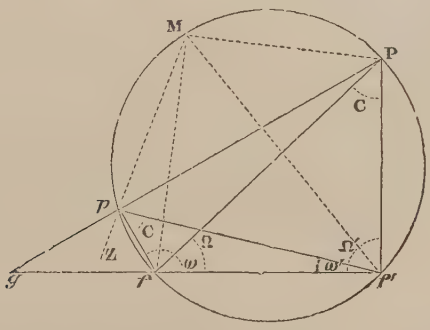
A circular disc of matter attracts as the inverse fifth power of the distance; a particle attracted by it may be made to describe any circle which cuts it orthogonally.

Now, as by Art. 6 any Cartesian oval is its own inverse with regard to the confocal circle which cuts it, it follows from Professor Sylvester's theory that if a disc in the form of a Cartesian oval attract as the inverse fifth power of the distance, a particle acted on by it may describe the confocal circle.

8. In the conics the curve is equally inclined to the lines drawn to the foci. I now proceed to prove the analogous theorem for the Cartesians:—*The arc of a Cartesian oval makes equal angles with the straight line drawn from the point to any focus, and the circular arc drawn from it through the two other foci.**

It is well known that if α, β be the angles which the tangent to the curve $\rho + m\rho' = 2a$ makes with the vectors, $\cos\alpha = m\cos\beta$.

Now, let f, f' be the two internal foci (of reference), draw any circle cutting the curve in P, p; Pp of course passing through the outer



focus g (not to complicate the figure, I have suppressed the curve itself): for convenience, throw the equation into the form

$$2c\frac{\sin\omega'}{\sin C} + 2cm\frac{\sin\omega}{\sin C} = 2a;$$

referring to the figure, we shall have then

$$\sin\omega' + m\sin\omega = \frac{a}{c}\sin C,$$

$$\sin\Omega' + m\sin\Omega = \frac{a}{c}\sin C;$$

$$\therefore \quad \sin\Omega' - \sin\omega' = m(\sin\omega - \sin\Omega),$$

or
$$\cos\frac{\Omega'+\omega'}{2}\sin\frac{\Omega'-\omega'}{2} = m\cos\frac{\omega+\Omega}{2}\sin\frac{\omega-\Omega}{2};$$

$$\therefore \quad \cos\frac{\Omega'+\omega'}{2} = m\cos\frac{\omega+\Omega}{2};$$

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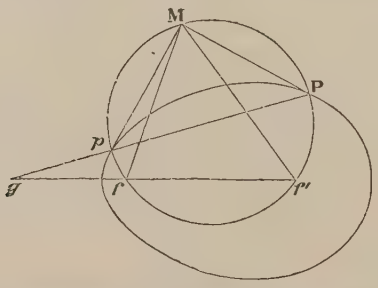
* It is obvious that the theorem in Art. 4 will also follow from this. This theorem was given in the *Educational Times* for March, 1866. In the April number of that journal, Mr. Clifford, of Trinity College, Cambridge, has published the more general theorem (with other remarkable properties)—that if 1, 2, 3, 4 are four concyclic foci of a bi-circular quartic, P any point on the curve, the arc of the curve is equally inclined to the circles P12, P34. It will be interesting to learn the method used by Mr. Clifford in arriving at these results.

i.e., bisecting the arc Pp in M, $\cos Mf'f = m \cos Mff'$. Join now Mp and produce it to Z, $Zpf = Mf'f$ and $Mpf' = Mff'$; hence $\cos Zpf = m \cos Mpf'$. Hence Mp is a tangent to the curve at p, and likewise MP is a tangent at P. Now Mp bisects the angle at p between the circle and its chord Pp. The theorem is proved then for a straight line to the outer focus and a circle to the two inner; and it follows easily, from elementary geometry, that, if true in this case, it is also true for the others—*e.g.*, for a straight line to f and a circle through gf'.

9. Hence, any circle being drawn through ff' cutting the curve in P, p, *the tangents at* P, p *are equal, and meet in* M, *the middle of the arc* Pp. As $\cos Mf'f = m \cos Mff'$, we see that the *locus of the intersection of pairs of equal tangents* is the locus of vertex of a triangle fMf', given the base and ratio of cosines of base angles. It is a curve of the fourth degree. Not only is this curve the locus for the Cartesian $\rho + m\rho' = 2a$, but for the family $\rho + m\rho' = 2A$, A being a variable parameter; which may be called a family of *parallel* Cartesians.



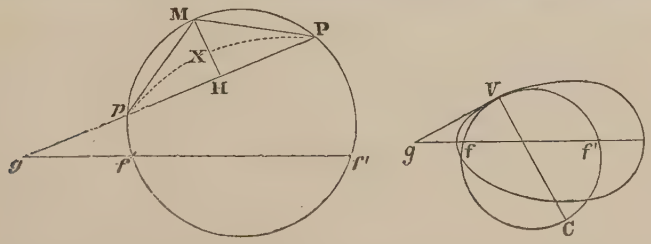
The locus of the intersection of pairs of equal normals is the circle $\rho = m\rho'$, and each pair are equally inclined to this circle.*

10. As the tangents MP, Mp are equal, a circle may always be drawn having double contact with the Cartesian at P and p. When the transversal gP becomes a tangent (touching the curve at the *apex*, Art. 7), this double-contact circle becomes in the limit the osculating circle at the apex, which therefore has contact of the third order

* The length of either normal is proportional to the vector from their common extremity to either focus. Hence an easy geometrical proof may be deduced of the theorem:—*The difference of the two arcs cut from any Cartesian by any pair of transversals from the external focus may be expressed by an elliptic function.* This may be considered as the complement of the theorem given by Mr. W. Roberts, of Trinity College, Dublin, as to the differences of the arcs of two conjugate ovals. It might doubtless, with proper precautions, be deduced by the method employed by Mr. Roberts; though, from the ambiguity of the polar equation, which he uses, it is difficult to know whether the two arcs we are considering belong to the same oval, or to two conjugates.—(See Salmon's *Higher Plane Curves.*)

(4-pointic). Hence the apex is the point of maximum or minimum curvature.

Let pXP be the arc of the double-contact circle at p, P; it is easy to see that, when pP becomes a tangent to the Cartesian, X is, in the



limit, the middle point of MH: hence the double-contact circle has its radius double of that of the circle Mff' at the limit. Hence, drawing a circle through ff' touching the curve at its apex V, and drawing its diameter VC, C is the centre of curvature at the apex.

11. Given a circle and an external point g, draw a tangent gV, and any arbitrary transversal gff'; a Cartesian is drawn, foci g, f, f' (g being external) to pass through V. The point V will be the common apex of all such Cartesians, and they have all 4-pointic contact there.*

12. A full discussion of the general equation of the Cartesian ovals, $\rho + m\rho' = 2a$, pointing out all the variations in form and position of the curve, corresponding to all possible changes in the signs and magnitudes of the constants m, a (and, particularly, when the curve is imaginary, when real), cannot be given here. There is difficulty in determining, in the vectorial system, whether a given point (ρ, ρ') is real; as, besides the obvious condition that negative values of ρ, ρ' are impossible, we must also have $\rho + \rho' > 2c$, and $\rho \sim \rho' < 2c$; or, which is the same thing, $\dfrac{\rho^2 + \rho'^2 - 4c}{2\rho\rho'}$ must be between the limits ± 1. Real values for the coordinates do not then necessarily give a real point. For instance, if we seek the intersections of the above curve with the

* Also if a fixed straight line and circle intersect in the points A and B, and any transversal be drawn cutting them in three points, a Cartesian can be drawn with those three points as foci, passing through A and B, and bisecting the angles made by the circle with the line; so that all such Cartesians have double contact with each other at A, B.

This is a case of a more general theorem by Professor Sylvester, in a recent number of the *Philosophical Magazine* obtained by a very different process, viz.: A circular cubic is cut by any transversal in three points; they will be the foci of a Cartesian oval passing through the four concyclic foci of the cubic. In the above, the circular cubic breaks up into a circle and straight line.

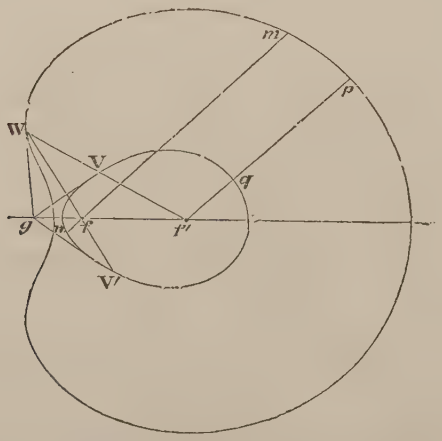
axis, we find that the axis is composed of three parts: ff', equation $\rho+\rho'=2c$; $f\infty$, equation $\rho'-\rho=2c$; $f'\infty$, equation $\rho-\rho'=2c$. Combining each of these in succession with $\rho+m\rho'=2c$, we should apparently find three intersections; but one will always be found imaginary, from negative values for ρ or ρ'; the two others are real. If a be intermediate in value between c and mc, the curve will always cut the line joining the foci ff', and will be a real curve.

Leaving, then, the full investigation, perhaps, for some future occasion, I would only state here one or two results at which I have arrived. It is obvious that generally the Cartesians can have no infinite branches; as, for an infinitely distant point, $\rho=\rho'$, which cannot be, unless $m=-1$, as in the hyperbola. Also, which is very important, it will be found that *one focus is always external and the other two internal.* They are always closed curves, and seem to present but two marked varieties of form (see Art. 13), which may be called *convex* Cartesians and *indented* Cartesians. If m and a are positive, the curve is always convex, but if one or both are negative, it may be convex or indented.

13. A pair of what are called *conjugate* ovals are related in a remarkable way. Such a pair would be (f, f' internal)

$$\left.\begin{aligned} m\rho'+\rho &= 2a \\ m\rho'-\rho &= 2a \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8),$$

where $m>1$, and $a>mc$. By the method in Art. 6, it is easy to show that, if from the focus f' any transversal $f'qp$ be drawn,



$f'p \cdot f'q = f'f \cdot f'g$. Hence *two conjugate ovals are inverse to each other with regard to the confocal circle whose centre is* f'.

Likewise, if a transversal be drawn through the *middle focus* f; $fm \,.\, fn = ff' \,.\, fg$.

For the outer focus g, we have seen that the rectangle of the segments of any transversal, made by the *same* oval, is constant.

These three important theorems as to the segments of transversals through the foci, are thus easily deduced from the vectorial equations. M. Quetelet, in his memoir on Caustics (*Mem. de Brux.*, tom v.), has been misled by the ambiguity of the polar equation, and concludes that if a radius vector be drawn from any focus cutting the two conjugate ovals, the rectangle of the segments is constant. This, we see, is only true for the extreme inner focus f'.

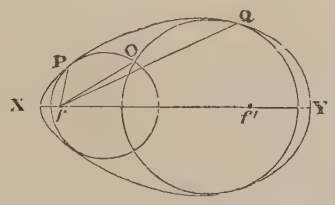
Any radius vector from f' cuts the two ovals in two points which are concyclic with g and f.

The points W, V (*apices*) are in directum with f', and concyclic with g, f.

The points W, V′ are in directum with f, and concyclic with g, f'.

14. Returning to the equation (7) of a double-contact circle to the Cartesian $\rho + m\rho' = 2a$,

$$\frac{\rho^2}{r} + m\frac{\rho'^2}{r'} = 2a,$$



we see that if (R, R′) be any point on the circle, $\left(\frac{R^2}{r}, \frac{R'^2}{r'}\right)$ will be a point on the Cartesian, and it is easy to show that the tangent circle at the latter point passes through (R, R′). Hence it follows that if two double-contact circles touch the curve at P, (r, r'), and Q, (s, s'); and if O, (R, R′), be their intersection, $fO^2 = fP \,.\, fQ$; or $R^2 = rs$; of course for the other two foci we shall also have $R'^2 = r's'$ and $R''^2 = r''s''$.

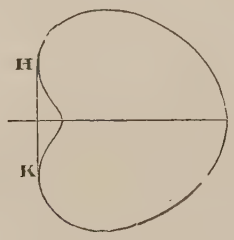
In equation (7), if (r, r') be one of the vertices X, Y of the curve, the tangent circle evidently becomes the osculating circle at the vertex, which has contact of the third order there. If, as in the figure, the two foci are internal, the osculating circles will be found to be

$$\left.\begin{aligned} \text{at X,}\quad & \frac{\rho^2}{a - cm} + m\frac{\rho'^2}{a + c} = \frac{4a}{m+1} \\ \text{at Y,}\quad & \frac{\rho^2}{a + cm} + m\frac{\rho'^2}{a - c} = \frac{4a}{m+1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (9).$$

The radii of curvature may easily be found by comparing with equation (6).

If, in (7), $\frac{1}{r} = -\frac{m}{r'}$, the tangent circle becomes the straight line

$$\rho^2 - \rho'^2 = 2ar.$$



This will be the *double tangent* HK to an *indented* Cartesian. Of course it is necessary that m be negative, as r, r' are positive; if m be positive the curve cannot be indented. Hence the locus of the points of contact H, K of the double tangent for all indented Cartesians of the *parallel* family $\rho - m\rho' = 2A$ will be the circle $\rho' = m\rho$.

Certain relations of the constants will cause one of the osculating circles in (9) to become a straight line, which will therefore meet the Cartesian in four consecutive points. The indentation in this case passes into a *point of undulation* at the vertex. One of the relations $a = \pm c \pm mc$ will be the condition required.

15. It is known that if T, T′ are the tangents from any point on an ellipse to two internal double-contact circles, $T + T' = C$. I have proved, by means of equation (7), that for a Cartesian, a linear relation holds between T and T′, $T + kT' = C$; but the limits of this paper oblige me to suppress many details; this proof, however, is sufficiently easy.

16. The following forms of the equation of the Cartesians seem remarkable, and promise interesting results:—It may be proved directly that if α, β, γ be any angles, the Cartesians

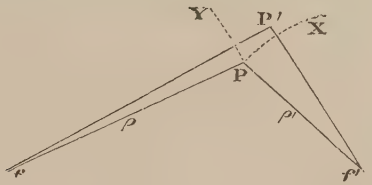
$$\pm\rho \sin\alpha \pm \rho' \sin\beta = 2c \sin\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

are cut orthogonally by the group

$$\pm\rho \cos\alpha \pm \rho' \cos\beta = 2c \cos\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11),$$

$2c$ being of course the distance ff' of the foci of reference. If, for instance, $\rho \sin\alpha + \rho' \sin\beta = 2c \sin\gamma$, and $\rho \cos\alpha - \rho' \cos\beta = 2c \cos\gamma$

are real, they intersect on the circle $\rho^2+\rho'^2-2\rho\rho'\cos(\alpha+\beta)=4c^2$. If, then, vectors be drawn to their intersection, the included angle is $\alpha+\beta$; and hence it is easily shown that the angles the tangents there make with either vector are complementary. A useful consideration here is that if P be a point $(\rho\rho')$, and P′ a point *in the*



vicinity $(\rho+\delta\rho,\ \rho'+\delta\rho')$, and if circles PX, PY are drawn with f', f as centres, ρ, ρ' as radii, then $\delta\rho, \delta\rho'$ are proportional to *the oblique coordinates of* P′ *referred to* PX, PY *as rectilinear axes.* These axes in this case contain an angle $\pi-(\alpha+\beta)$; and we easily see, from the formulas of Cartesian coordinates, with which we are familiar, that the tangents at P, viz., $\delta\rho\sin\alpha+\delta\rho'\sin\beta=0$, $\delta\rho\cos\alpha-\delta\rho'\cos\beta=0$, are at right angles.

We may show also that the curve $\rho\sin\alpha+\rho'\sin\beta=2c\sin\gamma$ *is cut at an angle* λ by the group

$$\rho\sin(\alpha\pm\lambda)+\rho'\sin(\beta\pm\lambda)=2c\sin(\gamma\pm\lambda) \quad \ldots\ldots\ldots \quad (12),$$

which cut it in the same points as the group (11).

From the expression (3) for c' it is easy to show that the curves (11) are triconfocal with (10), and it may be proved without difficulty by this method that triconfocal Cartesians cut orthogonally. The curves (10) and (11) belong to opposite series of the same confocal system.

It is worthy of remark that, as follows at once from equation (4), the equation of the group

$$\pm\rho\sin\alpha\pm\rho'\sin\beta=2c\sin\gamma$$

will be, when referred to the third focus g, and f, as in Art. 2,

$$\pm R\sin\beta\pm\rho\sin\gamma=2c'\sin\alpha.$$

17. Several interesting results will doubtless follow from the consideration of what may be termed a system of *parallel* Cartesians—the term, of course, merely referring to the form of the equation, viz.,

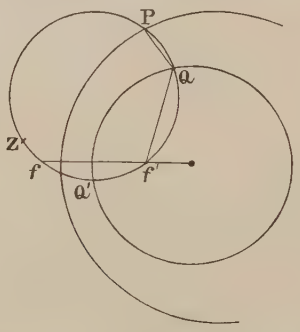
$$\rho=m\rho'+2A \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13).$$

A being an arbitrary parameter; the circle $\rho=m\rho'$ evidently bearing

a remarkable relation to the system of curves. I will here merely enumerate one or two.

1°. If any circle through f, f' cut a Cartesian $\rho = m\rho' + 2a$ and the parallel circle $\rho = m\rho'$ in two points P, Q as in the figure, QP is normal to the Cartesian at P; also $Qf' : QP$ in a constant ratio $(c : a)$.

2°. Hence if a circle through ff' cut the whole system of curves (13), the normals at all the intersections pass through a point (viz., Q, the



point where the circle meets $\rho = m\rho'$). For the intersections below ff', the normals all pass through Q'.

Also the tangents at all the intersections pass through a point (Z, the point diametrically opposite to Q).*

3°. If a pair of tangents be drawn from an *apex* V of a Cartesian to a *parallel* curve, they are equal in length; also the chord of contact is parallel to the tangent at V to the first curve.

4°. The locus of the *apices* of the system (13) is the curve in Art. 9.

18. It is worthy of notice that the proofs of certain theorems in rectangular coordinates will equally apply to the vectorial system; so that, by mere changes of the terms employed, we may deduce without labour many new and singular properties. Thus the equation of an axial circle being, as we have seen, of the same form as that of an ellipse in the rectangular system, we may at once conclude, from the known equation of a tangent to an ellipse, that the Cartesian touching the axial circle

$$A\rho^2 + B\rho'^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14).$$

at the point (rr'), must be

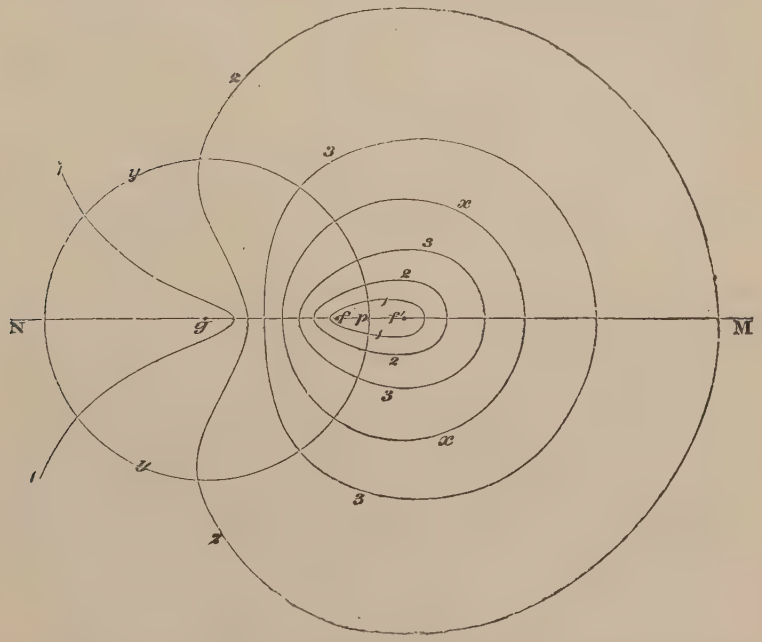
$$A\rho r + B\rho' r' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15).$$

* I find that the properties 1° and 2°, which are of importance in finding the caustic by refraction of a circle, are given by M. Quetelet (*loc. cit.*). No. 1° follows easily by considering that if D be the distance of two points (rr'), (s, s'), concyclic with f, f'; $2c\mathrm{D} = rs' - r's$.

If the point (rr') be outside the circle (14), then (15) represents the *Cartesian polar* of (rr'); *i.e.*, if Cartesians are drawn from (rr') touching the circle, it represents the Cartesian joining their points of contact. The theorems relating to poles and polars will then hold in this system. For instance, from every point on a Cartesian, pairs of tangent Cartesians are drawn to an axial circle; the Cartesian chord of contact always passes through a fixed point. Of course all the Cartesians spoken of are biconfocal.

From (15), it is easy to see that equation (7), $\frac{\rho^2}{r} + m\frac{\rho'^2}{r'} = 2a$, when the point (rr') is outside the curve, represents a circle such that the Cartesian tangents to it from (rr') meet it on the original curve $\rho + m\rho' = 2a$.

19. I have thought it might be useful to add a rough representation of a system of triconfocal Cartesians. In the figure, to avoid complication, I have only given one *series* of the system—that to which the



focus g is external; it is of course to be understood that there is an opposite *series* of like form, to which f' is the external focus, and which cut orthogonally the curves in the figure; of this opposite series, the

only one I have drawn is the confocal circle yy (centre g). The series given may again be subdivided into two—those *within* the confocal circle xx (centre f'), and those without, each of which is conjugate to one within. Thus the pairs marked respectively 11, 22, &c., are conjugate ovals. The outer curve 1, 1 is not completed for want of space. A pair of conjugates tends to two extreme limits—on the one hand, to coincide, when they fuse into the circle xx; on the other hand, the inner tends to become the line ff', the outer the infinite line gN. Of the ovals outside the circle xx, all are not *indented* (those nearest to the circle are convex), there being one intervening whose vertex is a point of undulation—neither convex nor concave. The whole series is cut orthogonally by the circle yy in the points I have termed their *apices*, or points of greatest or least curvature.

As I have already remarked, a systematic discussion of these interesting curves is still a desideratum; the first step would be to examine the variations of form and position of the curve which correspond to all possible changes in the constants of the vectorial equation; an investigation I hope possibly to continue hereafter. I should feel happy, however, if the few detached points I have laid before the Mathematical Society in the present paper had the effect of attracting the attention of more competent inquirers to the subject.*

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* I may indicate a graphic method of delineating, as it were, in a form familiar to us, the distinction between real and imaginary points in vectorial coordinates. Draw rectangular axes, OX, OY, and let $x = \rho$, $y = \rho'$; take on the positive axes two parts OA = OB = $2c$; erect two infinite perpendiculars to AB at the points A, B; viz., AM, BN (within the angle XOY). It will be easily seen that if (x, y) be within the infinite rectangle AMNB, $(\rho\rho')$ is a real point; if (x, y) be outside, $(\rho\rho')$ is impossible. The Cartesian $\rho + m\rho' = 2a$, then, will be real or imaginary according as the straight line $x + my = 2a$ does or does not intersect the above rectangle. This will easily give us, as the *conditions that the Cartesian* $\rho + m\rho' = 2a$ *be real*:—

1. If $\infty > m > 1$; then $a > c$.
2. If $1 > m > -1$; then $a > mc$.
3. If $-1 > m > -\infty$; then $a < c$.

If these are not fulfilled, the curve is imaginary. Of course, the above conditions can be otherwise found; but this construction presents to the mind in one *tableau* all the real and all the imaginary points. We see also from it, that even for a real curve, there are an infinite number of imaginary points satisfying its equation—corresponding to the parts of the line $x + my = 2a$ outside the infinite rectangle.

April 16, 1866.

Ordinary Meeting held at University College (Professor De Morgan in the Chair).

The following gentlemen were proposed and seconded for election as members at the next meeting of the Society:

Professor W. G. Adams, Messrs. O. G. Downes, and A. W. Young.

The following gentlemen were elected members of the Society:

Drs. Friedlander and Salmon, and Rev. R. Townsend.

The Paper read was—

"On the Correspondence between two Points on a Curve," by Professor Cayley.

Professor Hirst showed how Professor Cayley's method might be applied to the determination of the number of normals which can be drawn from a point to a curve.

Mr. T. Cotterill showed that on the sphere the analogue to the conchoid of Nicomedes and the limaçon of Pascal are the same curve.

Professor De Morgan gave a simple method of describing a small arc of a curve, when the two extreme points and the tangents at those points are given, by means of the closest parabola.

Professor Hirst made some remarks on quadric inversion.

May 21, 1866.

Ordinary Meeting held at University College (Professor de Morgan in the Chair).

The following gentlemen were proposed and seconded for election as members at the next meeting of the Society:

Rev. F. Harrison, Rev. G. S. Ward, Professor W. S. Jevons, Messrs. W. K. Clifford, W. Esson, F. H. Fisher, and I. Todhunter.

The following gentlemen were elected members of the Society:

Professor W. G. Adams, Messrs. O. G. Downes, and A. G. Young.

Professor Smith read a Paper "On the Multiplication of Four Theta Functions."

Professor De Morgan made some remarks on the method of quadratures, and gave a correction of the formula for the area of a curve when the arc is subdivided into four parts.

Mr. Jenkins gave a property of the periods of the reciprocals of composite numbers.

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April 16, 1866.

PROFESSOR DE MORGAN IN THE CHAIR.

On the Correspondence of Two Points on a Curve.

BY PROFESSOR CAYLEY, F.R.S.

1. IN a unicursal curve the coordinates (x, y, z) of any point of the curve are proportional to rational and integral functions of a variable parameter θ. Hence, if two points of the curve correspond in suchwise that to a given position of the first point there correspond α' positions of the second point, and to a given position of the second point α positions of the first point, the number of points which correspond each to itself is $= \alpha + \alpha'$. For let the two points be determined by their parameters θ, θ' respectively; then, to a given value of θ there correspond α values of θ', and to a given value of θ' there correspond α values of θ; hence the relation between (θ, θ') is of the form $(\theta, 1)^{\alpha}\,(\theta', 1)^{\alpha'} = 0$; and writing therein $\theta' = \theta$, then, for the points which correspond each to itself, we have an equation $(\theta, 1)^{\alpha+\alpha'} = 0$ of the order $\alpha + \alpha'$; that is, the number of these points is $= \alpha + \alpha$.

2. Hence, for a unicursal curve we have a theorem similar to that of M. Chasles' for a line; viz., the theorem may be thus stated:—

If two points of a unicursal curve have an (α, α') correspondence, the number of united points is $= \alpha + \alpha'$.

3. But a unicursal curve is nothing else than a curve with a deficiency $D = 0$, and we thence infer—

Theorem.—If two points of a curve with deficiency D have an (α, α') correspondence, the number of united points is $= \alpha + \alpha + 2k\mathrm{D}$; in which theorem $2k$ is a coefficient to be determined.

4. Suppose that the corresponding points are P, P′, and imagine that when P is given, the corresponding points P′ are the intersections of the given curve by a curve Θ (the equation of the curve Θ will of course contain the coordinates of P as parameters, for otherwise the

position of P′ would not depend upon that of P). I find that, if the curve Θ has with the given curve k intersections at the point P, then, in the system of (P, P′), the number of united points is

$$\mathfrak{a} = \alpha + \alpha' + 2k\mathrm{D};$$

whence, in particular, if the curve Θ does not pass through the point P, then the number of united points is $= \alpha + \alpha'$, as in a unicursal curve.

4.* The foregoing theorem is easily proved in the particular case where the k intersections at the point P take place in consequence of the curve θ having a k-tuple point at P. Taking $\mathrm{U} = (x, y, z)^m = 0$ as the equation of the given curve (which, for greater simplicity, is assumed to be a curve without singularities), then, if we suppose (x, y, z) to be the coordinates of the point P, and (x', y', z') to be the coordinates of the point P′, write $\mathrm{U} = (x, y, z)^m$, $\mathrm{U}' = (x', y', z')^m$, U' being what U becomes on writing therein (x', y', z') in place of (x, y, z); and

$$\Theta = (x, y, z)^a\ (x', y', z')^{a'}\ (yz' - y'z,\ zx' - z'x,\ xy' - x'y)^k;$$

viz., Θ is a function of the order k in $yz' - y'z$, $zx' - z'x$, $xy' - x'y$, the coefficients of the several powers and products of these quantities being functions of the order a in (x, y, z), and of the order a' in (x', y', z'), which functions are such that they do not all of them vanish, identically, or in virtue of the equation $\mathrm{U} = 0$, on writing therein $(x', y', z') = (x, y, z)$. Taking, for a moment, (x, y, z) as current coordinates, suppose that the equation of the given curve is $\mathrm{U} = 0$, then, if (x, y, z) are the coordinates of the point P, we have $\mathrm{U} = 0$, and similarly, if (x', y', z') are the coordinates of the point P′, we have $\mathrm{U}' = 0$. The equation $\Theta = 0$, considering therein (x, y, z) as the coordinates of the given point P (and so as parameters satisfying the equation $\mathrm{U} = 0$), and (x', y', z') as current coordinates, will be a curve having a k-tuple point at P—we have thus the case above supposed; and, P being given, the corresponding points P′ are given as the intersections of the curves $\mathrm{U}' = 0$, $\Theta = 0$, which are respectively of the orders m and $a' + k$; the total number of intersections is thus $= m\,(a' + k)$, but inasmuch as the curve $\Theta = 0$ has a k-tuple point at P, k of these intersections coincide with the point P, and the number of the remaining intersections, that is, the number of positions of the point P′, is $= ma' + (m-1)\,k$. Similarly, when P′ is given, the number of positions of the point P is $= ma + (m-1)\,k$; and we have, therefore,

$$\alpha + \alpha' = m\,(a+a') + 2\,(m-1)\,k.$$

To find the united points, it is to be observed, that upon writing $(x', y', z') = (x, y, z)$, the function Θ becomes identically $= 0$; but if we suppose, in the first instance, that P′, P are consecutive points on the

curve $U = 0$, then we have

$$yz' - y'z : zx' - z'x : xy' - x'y = \delta_x U, \delta_y U, \delta_z U;$$

and the equation $\Theta = 0$ assumes the form

$$\Theta = (x, y, z)^a \, (x, y, z)^{a'} \, (\delta_x U, \delta_y U, \delta_z U)^k = 0,$$

which, (x, y, z) being current coordinates, is the equation of a curve of the order $a + a' + (m-1)\,k$; the united points are the intersection of this curve with the given curve $U = 0$, and the number of the united points is thus

$$\mathfrak{a} = m\,(a + a') + m\,(m-1)\,k.$$

Hence, attending to the above-mentioned value of $a + a'$, we have

$$\mathfrak{a} = a + a' + (m-1)\,(m-2)\,k.$$

But in the case of a curve $U = 0$, without singularities, we have $2D = (m-1)\,(m-2)$, and we have thus the required formula

$$\mathfrak{a} = a + a' + 2kD.$$

The investigation in the case where the k intersections at P arise wholly or in part from a contact of the curve Θ, or any branch or branches thereof, with the given curve U, is more difficult, and I abstain from entering upon it.

I apply the theorem to some examples:—

5. Investigation of the class of a curve of the order m with δ dps:— Take as corresponding points on the curve two points, such that the line joining them passes through a fixed point O: the united points will be the points of contact of the tangents through O; that is, the number of the united points is equal to the class of the curve. The curve Θ is here the line OP, which has with the given curve a single intersection at P; that is, we have $k = 1$. The points P′ corresponding to a given position of P are the remaining $m-1$ intersections of OP with the curve; that is, we have $a' = m-1$; and, in like manner, $a = m-1$. Hence the class is $= 2\,(m-1) + 2D$, viz., this is $= (2m-2) + (m^2 - 3m + 2 - 2\delta)$, which is $= m^2 - m - 2\delta$, as it should be.

6. It is in the foregoing example assumed that the dps are none of them cusps; if the curve has $\delta + \kappa$ dps, κ of which are cusps (or, what is the same thing, δ nodes and κ cusps), then the number of united points is equal $2\,(m-1) + 2D$, $= m^2 - m - 2\delta - 2\kappa$; but, in this case, each of the cusps is reckoned as a united point, and we have, therefore, class $+ \kappa = m^2 - m - 2\delta - 2\kappa$, that is, class $= m^2 - m - 2\delta - 3\kappa$.

This will serve as an instance of the special considerations which are required in the case of a curve with cusps; but, in what follows, I shall assume that the dps are none of them cusps, and thus attend to the case of a curve of the order m, with δ dps, and therefore of the class $n = m^2 - m - 2\delta$, and of the deficiency $D = \frac{1}{2}(m-1)(m-2) - \delta$, $= \frac{1}{2}(n - 2m + 2)$.

7. Investigation of the number of inflexions:—Taking the point P to be a tangential of P (that is, an intersection of the curve by the tangent at P), the united points are the inflexions, and the number of the united points is equal to the number of inflexions. The curve Θ is here the tangent at P, having with the given curve two intersections at P; that is, $k = 2$. P′ is any one of the $m-2$ tangentials of P, hence $\alpha' = m - 2$; and P is the point of contact of any one of the $n-2$ tangents from P′ to the curve; that is, $\alpha = n - 2$. Hence the number of inflexions is $= (m-2) + (n-2) + 4D$, $= m + n - 4 + 2(n - 2m + 2)$, $= 3(n - m)$, which is right.

8. For the purpose of the next example it is necessary to present the fundamental equation $a = \alpha + \alpha' + 2kD$ under a more general form. The curve Θ may intersect the given curve in a system of points P′ each p times, a system of points Q′ each q times, &c., in such manner that the points (P, P′), the points (P, Q′), &c., are pairs of points corresponding to each other according to distinct laws; and we shall then have the numbers (a, α, α'), (b, β, β'), &c., belonging to these pairs respectively; viz., (P, P′) are points having an (α, α') correspondence, and the number of united points is $= a$; similarly (P, Q′) are points having a (β, β') correspondence, and the number of united points is $= b$; and so on. The theorem then is

$$p(a - \alpha - \alpha') + q(b - \beta - \beta') + \ldots = 2kD.$$

9. Investigation of the number of double tangents:—Take P′ an intersection with the curve of a tangent drawn from P to the curve (or, what is the same thing, P, P′ cotangentials of any point of the curve): the united points are here the points of contact of the several double tangents of the curve; or, if r be the number of double tangents, then the number of united points is $= 2r$. The curve Θ is the system of the $n-2$ tangents from P to the curve; each tangent has with the curve one intersection at P, that is, $k = n - 2$; each tangent, besides, meets the curve in the point of contact Q′ twice, and in $(m-3)$ points P′. Hence, if (a, α, α') refer to the points (P, Q′), and $(2r, \beta, \beta')$ to the points (P, P′), we have

$$2(a - \alpha - \alpha') + 2r - \beta - \beta' = 2(n-2)D.$$

Moreover, from the last example, the value of $a - \alpha - \alpha'$ is $= 4D$, and the formula thus becomes

$$2\tau - \beta - \beta' = 2\,(n-6)\,D;$$

but, from above, it appears that we have $\beta = \beta' = (n-2)\,(m-3)$, whence

$$\begin{aligned} 2\tau &= 2\,(n-2)\,(m-3) + 2\,(n-6)\,D, \\ &= 2\,(n-2)\,(m-3) + (n-6)\,(n-2m+2), \\ &= n^2 - 10n + 8m, \end{aligned}$$

which is right; in fact, observing that ι (the number of inflexions) is $= 3n - 3m$, the formula is equivalent to $2\tau + 3\iota = n^2 - n - m$, that is, $m = n^2 - n - 2\tau - 3\iota$.

In the foregoing examples, the curve Θ is a line or system of lines; but I give an example in which Θ is a system of conics, and in which, as will appear, we have to consider the two characteristics (μ, ν) of the system.

10. Investigation of the number of conics which can be drawn, satisfying any four conditions, and touching the given curve, or say of the number of the conics (4Z) (1):—Take P′ an intersection of the given curve by a conic (4Z) passing through the point P, then the number of the united points is equal to that of the conics (4Z) (1). The curve Θ is here the system of the conics (4Z) which pass through P; hence, if (μ, ν) be the characteristics of the system of conics (4Z), the number of the conics through P is $= \mu$; each of these has with the given curve one intersection at P, and consequently $k = \mu$. Moreover, each of the conics besides meets the curve in $(2m-1)$ points, and consequently $\alpha = \alpha' = \mu\,(2m-1)$. Hence the formula gives the number of united points

$$\begin{aligned} &= 2\mu\,(2m-1) + \mu\,(n-2m+2), \\ &= \mu\,(n+2m); \end{aligned}$$

or, as this may be written,

$$= \mu n + \nu m + m\,(2\mu - \nu).$$

But the system of conics (4Z) contains $(2\mu - \nu)$ point-pairs (*coniques infiniment aplaties*), each of which, regarded as a pair of coincident lines, meets the given curve in m pairs of coincident points; that is, the point-pair is to be considered as a conic *touching* the given curve in m points; and there is on this account a reduction $= m\,(2\mu - \nu)$ in the number of the united points; whence, finally, the number of the conics (4Z) (1) is $= \mu n + \nu m$. It is hardly necessary to remark that it is assumed that the conditions (4Z) are conditions having no special relation to the given curve.

11. As a final example, suppose that the point P on a given curve of the order m, and the point Q on a given curve of the order m', have an (a, a') correspondence, and let it be required to find the class of the curve enveloped by the line PQ. Take an arbitrary point O, join OQ, and let this meet the curve m in P', then (P, P') are points on the curve m, having a $(m'a, ma')$ correspondence; in fact, to a given position of P there correspond a' positions of Q, and to each of these m positions of P', that is, to each position of P, there correspond ma positions of P'; and similarly to each position of P' there correspond $m'a$ positions of P. The curve Θ is the system of the lines drawn from each of the a' positions of Q to the point O, hence the curve θ does not pass through P, and we have $k = 0$. Hence the number of the united points (P, P'), that is, the number of the lines PQ which pass through the point O, is $= ma' + m'a$, or this is the class of the curve enveloped by PQ.

13. It may be remarked, that if the two curves are curves in space (plane or of double curvature), then the like reasoning shows that the number of the lines PQ which meet a given line O is $= ma' + m'a$, that is, the order of the scroll generated by the line PQ is $= ma' + m'a$.

June 18, 1866.

Ordinary Meeting held at University College (Professor De Morgan in the Chair).

The following gentlemen were proposed and seconded for election as Members:—

Professors Thos. Barker and Price, Messrs. C. J. Faulkner, C. J. C. Price, and John Taylor.

The following gentlemen were elected Members:—

Rev. F. Harrison, Rev. G. S. Ward, Professor W. P. Jevons, Messrs. W. K. Clifford, F. H. Fisher, W. Esson, and I. Todhunter.

Mr. Spottiswoode brought before the Society the following problem:—

Given the total constituency of the country c, consisting of m Conservatives, and n Liberals; suppose that c is divided into p parts (*i.e.* counties, boroughs, &c.) $c_1, c_2, \ldots c_p$, in which the numbers of Conservatives and Liberals $(m_1, n_1, m_2, n_2, \ldots m_p, n_p)$ are unknown; required the probability of each party returning a majority in a House of Commons so elected. Also, how is that probability affected by increasing the number of, or by equalizing, the electoral districts.

The Author discussed the question in part, and exhibited, in the form of a table, the results which he had calculated for some special instances.

Mr. Thos. Cotterill gave some properties of cubic curves, with a method of tracing a circular cubic when one of the foci is on the curve.

June 26, 1866.

Special Meeting held at University College for the consideration of business (Professor De Morgan in the Chair).

The following gentlemen were elected Members:—

Professors T. Barker and Price, Messrs. C. J. Faulkner, C. J. C. Price, and John Taylor.

The following Resolution, proposed by Professor Hirst, and seconded by Mr. Spottiswoode, was agreed upon, viz.:—

"That the offer of the Chemical Society to lend their rooms at Burlington House to the Mathematical Society be accepted on the conditions made by them."

Professor Hirst presented a Paper containing a calculation by Mr. Shanks of the whole of the recurring period of the fraction $\frac{1}{17389}$.

Professor De Morgan gave a geometrical construction for determining the best straight line for approximating to the area of a curve.

*On a Formula for the Multiplication of four Theta Functions.**

By H. J. STEPHEN SMITH, M.A., F.R.S.,
Fellow of Balliol College, and Savilian Professor of Geometry in the University of Oxford.

Art. 1.—In a letter addressed to M. Hermite, and dated Berlin, Aug. 6, 1845,† Jacobi says:—

"Dans mes Leçons universitaires de Königsberg, moi aussi j'ai eu coutûme de partir des fonctions θ. Dans ces Leçons, en multipliant quatre séries $\sum_{-\infty}^{+\infty} \epsilon^{-(ax+bi)^2}$ pour différentes valeurs de x, et en transformant les exposants par la formule

$$i^2+i'^2+i''^2+i'''^2 = \left(\frac{i+i'+i''+i'''}{2}\right)^2+\left(\frac{i+i'-i''-i'''}{2}\right)^2+\ldots$$

j'ai obtenu tout de suite une formule de laquelle découlent, comme cas particuliers et sans le moindre calcul, les expressions fractionnaires des fonctions elliptiques, les théorèmes sur l'addition des trois espèces, et plusieurs centaines de formules intéressantes auxquelles on ne saurait arriver que par un calcul algébrique fatiguant."

The formula to which this passage refers has not (it would seem) been given by any writer on elliptic functions. The object of the present paper is to enunciate and demonstrate it; and to justify what Jacobi says of it by showing that many of the fundamental formulæ of the theory of elliptic functions are either particular cases of it, or corollaries from it. For the sake of symmetry, however, it is convenient to employ the arithmetical equality

$$i^2+i'^2+i''^2+i'''^2 = \left(\frac{-i+i'+i''+i'''}{2}\right)^2+\left(\frac{i-i'+i''+i'''}{2}\right)^2 + \left(\frac{i+i'-i''+i'''}{2}\right)^2+\left(\frac{i+i'+i''-i'''}{2}\right)^2,$$

instead of that indicated by Jacobi.

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* Read May 21, 1866. Professor De Morgan in the Chair.

† Jacobi, *Mathematische Werke*, Vol. I., p. 358.

ART. 2.—The Theta functions being defined by the equation

$$(1) \qquad \theta_{\mu,\mu'}(x) = \sum_{n=-\infty}^{n=+\infty} (-1)^{n\mu'} \epsilon^{i\pi\left[\frac{1}{2}(2n+\mu)^2\omega + (2n+\mu)\frac{x}{a}\right]},$$

(in which μ and μ' are given integral numbers, a any constant, ω an imaginary constant having for the coefficient of i* in its imaginary part a quantity different from zero and positive,) or, if $q = \epsilon^{i\pi\omega}$, $v = \epsilon^{\frac{i\pi}{a}}$, by the equation

$$(2) \qquad \theta_{\mu,\mu'}(x) = \sum_{n=-\infty}^{n=+\infty} (-1)^{n\mu'} q^{\frac{1}{2}(2n+\mu)^2} v^{(2n+\mu)x},$$

the formula for the multiplication of four Theta functions is

$$\begin{aligned}(3) \qquad & 2\theta_{\mu_1,\mu'_1}(x_1)\times\theta_{\mu_2,\mu'_2}(x_2)\times\theta_{\mu_3,\mu'_3}(x_3)\times\theta_{\mu_4,\mu'_4}(x_4) \\ &= \theta_{\sigma-\mu_1,\sigma'-\mu'_1}(s-x_1)\,\theta_{\sigma-\mu_2,\sigma'-\mu'_2}(s-x_2)\,\theta_{\sigma-\mu_3,\sigma'-\mu'_3}(s-x_3)\,\theta_{\sigma-\mu_4,\sigma'-\mu'_4}(s-x_4) \\ &+\theta_{\sigma-\mu_1,\sigma'-\mu'_1+1}(s-x_1)\,\theta_{\sigma-\mu_2,\sigma'-\mu'_2+1}(s-x_2)\,\theta_{\sigma-\mu_3,\sigma'-\mu'_3+1}(s-x_3)\,\theta_{\sigma-\mu_4,\sigma'-\mu'_4+1}(s-x_4) \\ &+(-1)^{\sigma'}\theta_{\sigma-\mu_1+1,\sigma'-\mu_1}(s-x_1)\,\theta_{\sigma-\mu_2+1,\sigma'-\mu'_2}(s-x_2)\,\theta_{\sigma-\mu_3+1,\sigma'-\mu'_3}(s-x_3) \\ &\qquad\qquad\times\theta_{\sigma-\mu_4+1,\sigma'-\mu'_4}(s-x_4) \\ &+(-1)^{\sigma'+1}\theta_{\sigma-\mu_1+1,\sigma'-\mu'_1+1}(s-x_1)\,\theta_{\sigma-\mu_2+1,\sigma'-\mu'_2+1}(s-x_2)\,\theta_{\sigma-\mu_3+1,\sigma'-\mu'_3+1}(s-x_3) \\ &\qquad\qquad\times\theta_{\sigma-\mu_4+1,\sigma'-\mu'_4+1}(s-x_4),\end{aligned}$$

where $2s = x_1+x_2+x_3+x_4$, $2\sigma = \mu_1+\mu_2+\mu_3+\mu_4$, $2\sigma' = \mu'_1+\mu'_2+\mu'_3+\mu'_4$, the sums $\mu_1+\mu_2+\mu_3+\mu_4$ and $\mu'_1+\mu'_2+\mu'_3+\mu'_4$ being supposed even.

Its proof depends on the two identities

$$(4) \quad \mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2 = (\sigma-\mu_1)^2+(\sigma-\mu_2)^2+(\sigma-\mu_3)^2+(\sigma-\mu_4)^2$$

$$\begin{aligned}(5) \quad \mu_1\mu'_1+\mu_2\mu'_2+\mu_3\mu'_3+\mu_4\mu'_4 &= (\sigma-\mu_1)(\sigma'-\mu'_1)+(\sigma-\mu_2)(\sigma'-\mu'_2) \\ &\quad +(\sigma-\mu_3)(\sigma'-\mu'_3)+(\sigma-\mu_4)(\sigma'-\mu'_4),\end{aligned}$$

both of which admit of immediate verification.

Representing the product

$$\theta_{\mu_1,\mu'_1}(x_1)\,\theta_{\mu_2,\mu'_2}(x_2)\,\theta_{\mu_3,\mu'_3}(x_3)\,\theta_{\mu_4,\mu'_4}(x_4)\ .$$

by Σ, we have, evidently,

$$\begin{aligned}(6) \quad 2\Sigma = 2\Sigma(-1)^{\mu'_1 n_1+\mu'_2 n_2+\mu'_3 n_3+\mu'_4 n_4} &\times q^{\frac{1}{2}[(2n_1+\mu_1)^2+(2n_2+\mu_2)^2+(2n_3+\mu_3)^2+(2n_4+\mu_4)^2]} \\ &\times v^{(2n_1+\mu_1)x_1+(2n_2+\mu_2)x_2+(2n_3+\mu_3)x_3+(2n_4+\mu_4)x_4},\end{aligned}$$

the sign of summation Σ extending to all integral values of n_1, n_2, n_3, n_4, from $-\infty$ to $+\infty$.

Let

$$(7) \qquad \begin{cases} N_1 = -n_1+n_2+n_3+n_4, \\ N_2 = \ \ n_1-n_2+n_3+n_4, \\ N_3 = \ \ n_1+n_2-n_3+n_4, \\ N_4 = \ \ n_1+n_2+n_3-n_4; \end{cases}$$

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* Here, and in the rest of this paper, i is used for $\sqrt{(-1)}$. The effect of the condition stated in the text is to render the analytical modulus of q inferior to unity, and thus ensure the convergence of the Theta series for all values of x real or imaginary.

so that, conversely,

$$(8)\qquad \begin{cases} 4n_1 = -N_1+N_2+N_3+N_4, \\ 4n_2 = \ \ N_1-N_2+N_3+N_4, \\ 4n_3 = \ \ N_1+N_2-N_3+N_4, \\ 4n_4 = \ \ N_1+N_2+N_3-N_4. \end{cases}$$

Transforming in (6) the index of q by the formula (4), and the indices of -1 and v by (5), we find

$$(9)\quad 2\Sigma = 2\Sigma\,(-1)^{\frac{1}{2}[N_1(\sigma'-\mu'_1)+N_2(\sigma'-\mu'_2)+N_3(\sigma'-\mu'_3)+N_4(\sigma'-\mu'_4)]} \times q^{\frac{1}{4}[(N_1+\sigma-\mu_1)^2+(N_2+\sigma-\mu_2)^2+(N_3+\sigma-\mu_3)^2+(N_4+\sigma-\mu_4)^2]} \times v^{(N_1+\sigma-\mu_1)(s-x_1)+(N_2+\sigma-\mu_2)(s-x_2)+(N_3+\sigma-\mu_3)(s-x_3)+(N_4+\sigma-\mu_4)(s-x_4)},$$

the sign of summation Σ extending to all values of N_1, N_2, N_3, N_4, defined by the equations (7); *i.e.*, to all integral values of N_1, N_2, N_3, N_4, from $-\infty$ to $+\infty$, which, substituted in the equations (8), give integral values to n_1, n_2, n_3, n_4. We have, therefore, to ascertain what values of N_1, N_2, N_3, N_4 satisfy this condition. In the first place, we see, from the equations (7), that the difference between any two of the four numbers N_1, N_2, N_3, N_4 is even; *i. e.*, that N_1, N_2, N_3, N_4 are all even or all uneven. (α) Let them be all even, and let $N_1=2\nu_1$, $N_2=2\nu_2$, $N_3=2\nu_3$, $N_4=2\nu_4$; substituting these values in the equations (8), we find that, in order to render n_1, n_2, n_3, n_4 integral, it is necessary and sufficient that $\nu_1+\nu_2+\nu_3+\nu_4$ should be even. (β) Let N_1, N_2, N_3, N_4 be all uneven, and let $N_1=2\nu_1+1$, $N_2=2\nu_2+1$, $N_3=2\nu_3+1$, $N_4=2\nu_4+1$; substituting as before, we find that, in order to render n_1, n_2, n_3, n_4 integral, it is necessary and sufficient that $\nu_1+\nu_2+\nu_3+\nu_4$ should be uneven. Separating the terms in (9), in which N_1, N_2, N_3, N_4 are all even, from those in which N_1, N_2, N_3, N_4 are all uneven, we obtain

$$(10)\quad \begin{aligned} 2\Sigma = {} & 2\Sigma'\,(-1)^{\nu_1(\sigma'-\mu'_1)+\nu_2(\sigma'-\mu'_2)+\nu_3(\sigma'-\mu'_3)+\nu_4(\sigma'-\mu'_4)} \\ & \times q^{\frac{1}{4}[(2\nu_1+\sigma-\mu_1)^2+(2\nu_2+\sigma-\mu_2)^2+(2\nu_3+\sigma-\mu_3)^2+(2\nu_4+\sigma-\mu_4)^2]} \\ & \times v^{(2\nu_1+\sigma-\mu_1)(s-x_1)+(2\nu_2+\sigma-\mu_2)(s-x_2)+(2\nu_3+\sigma-\mu_3)(s-x_3)+(2\nu_4+\sigma-\mu_4)(s-x_4)} \\ & +2\,(-1)^{\sigma}\,\Sigma''\,(-1)^{\nu_1(\sigma'-\mu'_1)+\nu_2(\sigma'-\mu'_2)+\nu_3(\sigma'-\mu'_3)+\nu_4(\sigma'-\mu'_4)} \\ & \times q^{\frac{1}{4}[(2\nu_1+\sigma-\mu_1+1)^2+(2\nu_2+\sigma-\mu_2+1)^2+(2\nu_3+\sigma-\mu_3+1)^2+(2\nu_4+\sigma-\mu_4+1)^2]} \\ & \times v^{(2\nu_1+\sigma-\mu_1+1)(s-x_1)+(2\nu_2+\sigma-\mu_2+1)(s-x_2)+(2\nu_3+\sigma-\mu_3+1)(s-x_3)+(2\nu_4+\sigma-\mu_4+1)(s-x_4)}, \end{aligned}$$

the signs of summation Σ' and Σ'' referring to all values of ν_1, ν_2, ν_3, ν_4 from $-\infty$ to $+\infty$, for which the sum $\nu_1+\nu_2+\nu_3+\nu_4$ is even and uneven respectively. If, for a moment, we represent the general terms of the series Σ' and Σ'' by P and Q, we have, evidently,

$$2\Sigma'.\,P = \Sigma\,.\,P+\Sigma\,(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}P, \quad 2\Sigma''.\,Q = \Sigma\,.\,Q-\Sigma\,(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}Q,$$

the signs of summation Σ extending to all values of ν_1, ν_2, ν_3, ν_4 from $-\infty$ to $+\infty$ without any limitation. Substituting these values in

(10), we obtain, finally,

$$(11)\qquad 2\Sigma = \Sigma . P + \Sigma (-1)^{\nu_1+\nu_2+\nu_3+\nu_4} P + (-1)^{\sigma'} \Sigma . Q + (-1)^{\sigma'+1} \Sigma (-1)^{\nu_1+\nu_2+\nu_3+\nu_4} Q,$$

an equation which, on replacing P and Q by the expressions which they represent, will be found to coincide with (3).

ART. 3. — We shall now develope some of the particular results included in the formula (3): in doing so, we shall have occasion to employ the equations

$$(12)\qquad \theta_{\mu+2,\mu'}(x) = (-1)^{\mu'} \theta_{\mu,\mu'}(x); \quad \theta_{\mu,\mu'+2}(x) = \theta_{\mu,\mu'}(x),$$

$$(13)\qquad \theta_{\mu,\mu'}(-x) = (-1)^{\mu\mu'} \theta_{\mu,\mu'}(x),$$

which are immediate consequences of the equation of definition (1) or (2). From (12) we infer that there are only four distinct Theta functions, $\theta_{0,0}(x)$, $\theta_{0,1}(x)$, $\theta_{1,0}(x)$, $\theta_{1,1}(x)$; from (13) we infer that the first three of these are even functions, the last an uneven function; so that $\theta_{1,1}(0)$, and the three derived functions $\theta'_{0,0}(0)$, $\theta'_{0,1}(0)$, $\theta'_{1,0}(0)$ are all zero.

We successively attribute to the symbols

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ \mu_1, & \mu_2, & \mu_3, & \mu_4 \\ \mu'_1, & \mu'_2, & \mu'_3, & \mu'_4 \end{vmatrix}$$

the systems of values

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} x, & x, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} x, & x, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 1, & 1, & 0, & 0 \end{vmatrix},$$

$$\begin{vmatrix} x, & x, & 0, & 0 \\ 1, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix}, \quad \begin{vmatrix} x, & x, & 0, & 0 \\ 1, & 1, & 0, & 0 \\ 1, & 1, & 0, & 0 \end{vmatrix};$$

and we obtain the equations

$$(14)\qquad \theta^4_{0,0}(0) = \theta^4_{0,1}(0) + \theta^4_{1,0}(0),$$

$$(15)\qquad \begin{cases} \theta^2_{0,0}(x)\,\theta^2_{0,0}(0) = \theta^2_{0,1}(x)\,\theta^2_{0,1}(0) + \theta^2_{1,0}(x)\,\theta^2_{1,0}(0), \\ \theta^2_{0,1}(x)\,\theta^2_{0,0}(0) = \theta^2_{0,0}(x)\,\theta^2_{0,1}(0) - \theta^2_{1,1}(x)\,\theta^2_{1,0}(0), \\ \theta^2_{1,0}(x)\,\theta^2_{0,0}(0) = \theta^2_{1,1}(x)\,\theta^2_{0,1}(0) + \theta^2_{0,0}(x)\,\theta^2_{1,0}(0), \\ \theta^2_{1,1}(x)\,\theta^2_{0,0}(0) = \theta^2_{1,0}(x)\,\theta^2_{0,1}(0) - \theta^2_{0,1}(x)\,\theta^2_{1,0}(0), \end{cases}$$

of which (14) arises from the first of (15) by putting 0 for x; and the four equations (15) are equivalent to two independent equations.

ART. 4.—Again, attributing to the elements x_1, x_2, x_3, x_4 the values $x-y$, $x+y$, 0, 0, we may obtain formulæ expressing the product of the

Theta functions of the sum and difference of two quantities in terms of the Theta functions of the two quantities themselves. Thus, to obtain the simplest possible expressions for the six products

$$\theta_{0,0}(x-y)\,\theta_{0,1}(x+y),\quad \theta_{0,0}(x-y)\,\theta_{1,0}(x+y),\quad \theta_{0,0}(x-y)\,\theta_{1,1}(x+y),$$
$$\theta_{0,1}(x-y)\,\theta_{1,0}(x+y),\quad \theta_{0,1}(x-y)\,\theta_{1,1}(x+y),\quad \theta_{1,0}(x-y)\,\theta_{1,1}(x+y)$$

(in which the two Theta functions are different), we attribute to the elements

$$\left|\begin{matrix}\mu_1, & \mu_2, & \mu_3, & \mu_4\\ \mu_1', & \mu_2', & \mu_3', & \mu_4'\end{matrix}\right|$$

the systems of values

$$\left|\begin{matrix}0, & 0, & 0, & 0\\ 0, & 1, & 0, & 1\end{matrix}\right|,\quad \left|\begin{matrix}0, & 1, & 0, & 1\\ 0, & 0, & 0, & 0\end{matrix}\right|,\quad \left|\begin{matrix}0, & 1, & 0, & 1\\ 0, & 1, & 1, & 0\end{matrix}\right|,$$
$$\left|\begin{matrix}0, & 1, & 0, & 1\\ 1, & 0, & 1, & 0\end{matrix}\right|,\quad \left|\begin{matrix}0, & 1, & 0, & 1\\ 1, & 1, & 0, & 0\end{matrix}\right|,\quad \left|\begin{matrix}1, & 1, & 0, & 0\\ 0, & 1, & 0, & 1\end{matrix}\right|;$$

and we obtain the results

(16) $$\theta_{0,0}(x-y)\,\theta_{0,1}(x+y)\,\theta_{0,0}(0)\,\theta_{0,1}(0)$$
$$=\theta_{0,1}(y)\,\theta_{0,0}(y)\,\theta_{0,1}(x)\,\theta_{0,0}(x)-\theta_{1,1}(y)\,\theta_{1,0}(y)\,\theta_{1,1}(x)\,\theta_{1,0}(x),$$

(17) $$\theta_{0,0}(x-y)\,\theta_{1,0}(x+y)\,\theta_{0,0}(0)\,\theta_{1,0}(0)$$
$$=\theta_{1,0}(y)\,\theta_{0,0}(y)\,\theta_{1,0}(x)\,\theta_{0,0}(x)+\theta_{1,1}(y)\,\theta_{0,1}(y)\,\theta_{1,1}(x)\,\theta_{0,1}(x),$$

(18) $$\theta_{0,0}(x-y)\,\theta_{1,1}(x+y)\,\theta_{0,1}(0)\,\theta_{1,0}(0)$$
$$=\theta_{1,1}(y)\,\theta_{0,0}(y)\,\theta_{1,0}(x)\,\theta_{0,1}(x)+\theta_{1,0}(y)\,\theta_{0,1}(y)\,\theta_{1,1}(x)\,\theta_{0,0}(x),$$

(19) $$\theta_{0,1}(x-y)\,\theta_{1,0}(x+y)\,\theta_{0,1}(0)\,\theta_{1,0}(0)$$
$$=\theta_{1,0}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,0}(x),$$

(20) $$\theta_{0,1}(x-y)\,\theta_{1,1}(x+y)\,\theta_{0,0}(0)\,\theta_{1,0}(0)$$
$$=\theta_{1,0}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,0}(x),$$

(21) $$\theta_{1,0}(x-y)\,\theta_{1,1}(x+y)\,\theta_{0,0}(0)\,\theta_{0,1}(0)$$
$$=\theta_{0,1}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{1,0}(x)+\theta_{1,1}(y)\,\theta_{1,0}(y)\,\theta_{0,1}(x)\,\theta_{0,0}(x),$$

of which the number may be doubled by changing $+y$ into $-y$.

It will be observed that the values attributed to $\left|\begin{matrix}\mu_1, & \mu_2\\ \mu_1', & \mu_2'\end{matrix}\right|$ are determined in each of these equations by the indices of the Theta functions in the given product $\theta_{\mu_1,\mu_1'}(x-y)\,\theta_{\mu_2,\mu_2'}(x+y)$: the values of $\left|\begin{matrix}\mu_3, & \mu_4\\ \mu_3', & \mu_4'\end{matrix}\right|$ are then determined by the conditions that 2σ and $2\sigma'$ must be even, and that, since $\theta_{1,1}(0)=0$, the combination $\left|\begin{matrix}1\\ 1\end{matrix}\right|$ is inadmissible.

ART. 5.—If we differentiate the formulæ (16)—(21) with respect to y, and then put $y=0$, we arrive at the following equations, which serve to express the differential coefficient of a quotient of two Theta functions, in terms of the Theta functions themselves:—

$$(22)\quad \theta_{0,1}(x)\,\theta'_{0,0}(x)-\theta_{0,0}(x)\,\theta'_{0,1}(x)=\frac{\theta'_{1,1}(0)\,\theta_{1,0}(0)}{\theta_{0,0}(0)\,\theta_{0,1}(0)}\,\theta_{1,1}(x)\,\theta_{1,0}(x),$$

$$(23)\quad \theta_{1,0}(x)\,\theta'_{0,0}(x)-\theta_{0,0}(x)\,\theta'_{1,0}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,1}(0)}{\theta_{0,0}(0)\,\theta_{1,0}(0)}\,\theta_{1,1}(x)\,\theta_{0,1}(x),$$

$$(24)\quad \theta_{1,1}(x)\,\theta'_{0,0}(x)-\theta_{0,0}(x)\,\theta'_{1,1}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,0}(0)}{\theta_{0,1}(0)\,\theta_{1,0}(0)}\,\theta_{1,0}(x)\,\theta_{0,1}(x),$$

$$(25)\quad \theta_{1,0}(x)\,\theta'_{0,1}(x)-\theta_{0,1}(x)\,\theta'_{1,0}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,0}(0)}{\theta_{0,1}(0)\,\theta_{1,0}(0)}\,\theta_{1,1}(x)\,\theta_{0,0}(x),$$

$$(26)\quad \theta_{1,1}(x)\,\theta'_{0,1}(x)-\theta_{0,1}(x)\,\theta'_{1,1}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{0,1}(0)}{\theta_{0,0}(0)\,\theta_{1,0}(0)}\,\theta_{1,0}(x)\,\theta_{0,0}(x),$$

$$(27)\quad \theta_{1,1}(x)\,\theta'_{1,0}(x)-\theta_{1,0}(x)\,\theta'_{1,1}(x)=-\frac{\theta'_{1,1}(0)\,\theta_{1,0}(0)}{\theta_{0,0}(0)\,\theta_{0,1}(0)}\,\theta_{0,1}(x)\,\theta_{0,0}(x).$$

ART. 6.—Each of the four products

$$\theta_{0,0}(x-y)\,\theta_{0,0}(x+y),\quad \theta_{0,1}(x-y)\,\theta_{0,1}(x+y),\quad \theta_{1,0}(x-y)\,\theta_{1,0}(x+y),\quad \theta_{1,1}(x-y)\,\theta_{1,1}(x+y)$$

(in which the two Theta functions are the same) can be expressed in six different ways in terms of the squares of the Theta functions of x and y: thus

$$(28)\quad \theta_{\mu,\mu'}(x-y)\,\theta_{\mu,\mu'}(x+y)\,\theta^2_{0,0}(0)$$
$$=\theta^2_{0,1}(y)\,\theta^2_{\mu,\mu'+1}(x)+(-1)^{\mu'}\theta^2_{1,0}(y)\,\theta^2_{\mu+1,\mu'}(x)$$
$$=\theta^2_{0,0}(y)\,\theta^2_{\mu,\mu'}(x)+(-1)^{\mu'}\theta^2_{1,1}(y)\,\theta^2_{\mu+1,\mu'+1}(x),$$

$$(29)\quad \theta_{\mu,\mu'}(x-y)\,\theta_{\mu,\mu'}(x+y)\,\theta^2_{0,1}(0)$$
$$=\theta^2_{0,0}(y)\,\theta^2_{\mu,\mu'+1}(x)+(-1)^{\mu'}\theta^2_{1,0}(y)\,\theta^2_{\mu+1,\mu'+1}(x)$$
$$=\theta^2_{0,1}(y)\,\theta^2_{\mu,\mu'}(x)+(-1)^{\mu'}\theta^2_{1,1}(y)\,\theta^2_{\mu+1,\mu'}(x),$$

$$(30)\quad \theta_{\mu,\mu'}(x-y)\,\theta_{\mu,\mu'}(x+y)\,\theta^2_{1,0}(0)$$
$$=(-1)^{\mu'}\theta^2_{0,0}(y)\,\theta^2_{\mu+1,\mu'}(x)-(-1)^{\mu'}\theta^2_{0,1}(y)\,\theta^2_{\mu+1,\mu'+1}(x)$$
$$=\theta^2_{1,0}(y)\,\theta^2_{\mu,\mu'}(x)-\theta^2_{1,1}(y)\,\theta^2_{\mu,\mu'+1}(x).$$

To form these equations, we represent by (A), (A′), (B), (B′), (C), (C′), the formulæ obtained by attributing to the indices in (3) the values

$$(A)\quad \begin{vmatrix} \mu, & \mu, & 0, & 0 \\ \mu', & \mu', & 0, & 0 \end{vmatrix},\qquad (A')\quad \begin{vmatrix} \mu+1, & \mu+1, & 1, & 1 \\ \mu'+1, & \mu'+1, & 1, & 1 \end{vmatrix},$$

$$(B)\quad \begin{vmatrix} \mu, & \mu, & 0, & 0 \\ \mu', & \mu', & 1, & 1 \end{vmatrix}, \qquad (B')\quad \begin{vmatrix} \mu+1, & \mu+1, & 1, & 1 \\ \mu', & \mu', & 1, & 1 \end{vmatrix},$$

$$(C)\quad \begin{vmatrix} \mu, & \mu, & 1, & 1 \\ \mu', & \mu', & 0, & 0 \end{vmatrix}, \qquad (C')\quad \begin{vmatrix} \mu, & \mu, & 1, & 1 \\ \mu'+1, & \mu'+1, & 1, & 1 \end{vmatrix};$$

the indeterminates x_1, x_2, x_3, x_4 receiving the values $x-y$, $x+y$, 0, 0: the left-hand members of the three formulæ (A′), (B′), (C′) contain the factor $\theta_{1,1}^2(0)$, and are therefore zero; and the equations (A) ± (A′), (B) ± (B′), (C) ± (C′) will be found to coincide respectively with (28), (29), (30).*

One of the formulæ (29), by putting $\mu=0$, $\mu'=1$, becomes

$$(31)\quad \theta_{0,1}(x-y)\,\theta_{0,1}(x+y)\,\theta_{0,1}^2(0) = \theta_{0,1}^2(y)\,\theta_{0,1}^2(x) - \theta_{1,1}^2(y)\,\theta_{1,1}^2(x),$$

an equation which is particularly important in the theory of elliptic functions.

Art. 7.—Differentiating the equations (28)—(30) twice with respect to y, and putting $y=0$, we obtain a system of equations, of which the following, derived from (31), is one:

$$(32)\quad \frac{d}{dx}\left[\frac{\theta'_{0,1}(x)}{\theta_{0,1}(x)}\right] = \frac{\theta''_{0,1}(0)}{\theta_{0,0}(0)} - \frac{\theta_{1,1}^2(0)\,\theta_{1,1}^2(x)}{\theta_{0,1}^2(0)\,\theta_{0,1}^2(x)}.$$

If in certain of the equations of this system† we put $x=0$; or, more simply, if we differentiate the equations (15) twice with respect to x, and then put $x=0$, we find

$$(33)\quad \begin{cases} \theta_{0,0}^3\,\theta''_{0,0} = \theta_{0,1}^3\,\theta''_{0,1} + \theta_{1,0}^3\,\theta''_{1,0}, \\ \theta_{0,0}^2\,\theta_{1,1}'^2 = \theta_{0,1}^2\,\theta_{1,0}\,\theta''_{1,0} - \theta_{1,0}^2\,\theta_{0,1}\,\theta''_{0,1}, \\ \theta_{0,1}^2\,\theta_{1,1}'^2 = \theta_{0,0}^2\,\theta_{1,0}\,\theta''_{1,0} - \theta_{1,0}^2\,\theta_{0,0}\,\theta''_{0,0}, \\ \theta_{1,0}^2\,\theta_{1,1}'^2 = \theta_{0,1}^2\,\theta_{0,0}\,\theta''_{0,0} - \theta_{0,0}^2\,\theta_{0,1}\,\theta''_{0,1}, \end{cases}$$

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* All the equations (16)—(21), and some of the equations (28)—(30), will be found in an excellent memoir of M. Betti, "La Teorica delle funzione ellittiche e sue applicazione" ("Tortolini," Vol. III.; New Series, p. 126). M. Betti derives these equations from the formula

$$\begin{aligned} 2\theta_{\mu,\nu}(x+y)\,\theta_{\mu',\nu'}(x-y)\,\theta_{\alpha,0}(0)\,\theta_{0,\beta}(0) \\ = \theta_{\mu,\nu}(x)\,\theta_{\mu',\nu'}(x)\,\theta_{\alpha,0}(y)\,\theta_{0,\beta}(y) + (-1)^{\nu}\,\theta_{\mu+1,\nu}(x)\,\theta_{\mu'+1,\nu'}(x)\,\theta_{\alpha+1,0}(y)\,\theta_{1,\beta}(y) \\ + (-1)^{\nu}\,\theta_{\mu+1,\nu+1}(x)\,\theta_{\mu'+1,\nu'+1}(x)\,\theta_{\alpha+1,1}(y)\,\theta_{1,\beta+1}(y) \\ + \theta_{\mu,\nu+1}(x)\,\theta_{\mu',\nu'+1}(x)\,\theta_{\alpha,1}(y)\,\theta_{0,\beta+1}(y), \end{aligned}$$

where $\alpha=\mu-\mu'$, $\beta=\nu'-\nu$; which is a particular case of the formula (3), and which has also been given by M. Hermite ("Liouville," New Series, Vol. III., p. 27). The notation of M. Betti is different from that employed here, and by M. Hermite in the memoir just cited.

† Viz.: in (28) let $\mu=\mu'=0$, and in (28), (29), (30) let $\mu=\mu'=1$.

where $\theta_{0,0}$, $\theta''_{0,0}$, &c. are written for $\theta_{0,0}(0)$, $\theta''_{0,0}(0)$, &c. This system is equivalent to two independent relations between $\theta''_{0,0}$, $\theta''_{0,1}$, and $\theta''_{1,0}$.

ART. 8.—Lastly, in the equation (3), let $x_1+x_2+x_3=0$, $\mu_1=\mu_2=\mu_3=\mu_4=\mu$, $\mu'_1=\mu'_2=\mu'_3=\mu'_4=\nu$; differentiate the equation with respect to x_4, and afterwards put $x_4=0$: we find, on combining with one another (by addition or subtraction) the three equations, answering to the three combiuations $\mu=0$, $\nu=0$; $\mu=0$, $\nu=1$; $\mu=1$, $\nu=0$,

$$(34)\quad \theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3)\,\theta'_{1,1}(0) = (-1)^{\mu+\nu}\theta_{\mu,\nu}(0)\,[\theta_{\mu,\nu}(x_1)\,\theta_{\mu,\nu}(x_2)\,\theta'_{\mu,\nu}(x_3) + \theta_{\mu,\nu}(x_1)\,\theta'_{\mu,\nu}(x_2)\,\theta_{\mu,\nu}(x_3) + \theta'_{\mu,\nu}(x_1)\,\theta_{\mu,\nu}(x_2)\,\theta_{\mu,\nu}(x_3)],$$

where, on the right-hand side, μ and ν may have any one of the three sets of values $\mu=0$, $\nu=0$; $\mu=0$, $\nu=1$; $\mu=1$, $\nu=1$.

ART. 9.—It remains to apply these formulæ to the demonstration of the fundamental properties of elliptic functions.

If $\sqrt{k}$, $\sqrt{k'}$ represent the fractions $\frac{\theta_{1,0}(0)}{\theta_{0,0}(0)}$, $\frac{\theta_{0,1}(0)}{\theta_{0,0}(0)}$, the three elliptic functions of the first species are defined by the equations

$$(35)\qquad \sin am\,x = \frac{1}{\sqrt{k}}\cdot\frac{1}{i}\cdot\frac{\theta_{1,1}(x)}{\theta_{0,1}(x)},$$

$$(36)\qquad \cos am\,x = \frac{\sqrt{k'}}{\sqrt{k}}\cdot\frac{\theta_{1,0}(x)}{\theta_{0,1}(x)},$$

$$(37)\qquad \Delta\, am\,x = \sqrt{k'}\cdot\frac{\theta_{0,0}(x)}{\theta_{0,1}(x)},$$

in which the constant a of equation (1) is to receive a certain value which will presently be assigned. Introducing these functions into the equations (14) and (15), we find

$$(38)\qquad k^2+k'^2=1,$$

$$(39)\qquad \cos^2 am\,x+\sin^2 am\,x=1,$$

$$(40)\qquad \Delta^2 am\,x+k^2\sin^2 am\,x=1.$$

Again, the differential equation (26) assumes the form

$$\frac{d\,.\,\sin am\,x}{dx}=\frac{1}{i}\,\frac{\theta_{0,0}(0)\,\theta'_{1,1}(0)}{\theta_{1,0}(0)\,\theta_{0,1}(0)}\cos am\,x\,\Delta\, am\,x.$$

To identify the function $\sin am\,x$ with the function u which satisfies the differential equation

$$\frac{du^2}{dx^2}=(1-u^2)(1-k^2u^2),$$

and the initial conditions $u=0$, $\frac{du}{dx}=+1$, when $x=0$, it is necessary and sufficient that the coefficient $\frac{1}{i}\frac{\theta_{0,0}(0)\,\theta'_{1,1}(0)}{\theta_{1,0}(0)\,\theta_{0,1}(0)}$ should be equal to unity. This condition is satisfied by assigning to the constant a, which has remained undetermined in the equations (1) and (2), the value

$$\pi \times \frac{\sum_{-\infty}^{+\infty} q^{n^2} \times \sum_{-\infty}^{+\infty} (-1)^n (2n+1)\, q^{\frac{1}{4}(2n+1)^2}}{\sum_{-\infty}^{+\infty} (-1)^n q^{n^2} \times \sum_{-\infty}^{+\infty} q^{\frac{1}{4}(2n+1)^2}}.$$

Adopting this value of the constant a, we obtain from (26), (25), and (22),

$$\frac{d\,.\,\sin am\,x}{dx} = \cos am\,x\,\Delta am\,x, \tag{41}$$

$$\frac{d\,.\,\cos am\,x}{dx} = -\sin am\,x\,\Delta am\,x, \tag{42}$$

$$\frac{d\,.\,\Delta am\,x}{dx} = -k^2 \sin am\,x \cos am\,x. \tag{43}$$

Next, dividing the equations (29), (19), and (16) by equation (31), and changing y into $-y$ in (16), we find

$$\frac{\theta_{0,0}(0)\,\theta_{1,0}(0)}{\theta^2_{0,1}(0)}\,\frac{\theta_{1,1}(x+y)}{\theta_{0,1}(x+y)} = \frac{\theta_{1,0}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,0}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)-\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)},$$

$$\frac{\theta_{1,0}(0)}{\theta_{0,1}(0)}\,\frac{\theta_{1,0}(x+y)}{\theta_{0,1}(x+y)} = \frac{\theta_{1,0}(y)\,\theta_{0,1}(y)\,\theta_{1,0}(x)\,\theta_{0,1}(x)+\theta_{1,1}(y)\,\theta_{0,0}(y)\,\theta_{1,1}(x)\,\theta_{0,0}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)-\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)},$$

$$\frac{\theta_{0,0}(0)}{\theta_{0,1}(0)}\,\frac{\theta_{0,0}(x+y)}{\theta_{0,1}(x+y)} = \frac{\theta_{0,1}(y)\,\theta_{0,0}(y)\,\theta_{0,1}(x)\,\theta_{0,0}(x)+\theta_{1,1}(y)\,\theta_{1,0}(y)\,\theta_{1,1}(x)\,\theta_{1,0}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)-\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)};$$

or, introducing the elliptic functions

$$\sin am\,(x+y) = \frac{\cos am\,y\,\Delta am\,y \sin am\,x+\cos am\,x\,\Delta am\,x \sin am\,y}{1-k^2 \sin^2 am\,x \sin^2 am\,y}, \tag{44}$$

$$\cos am\,(x+y) = \frac{\cos am\,x \cos am\,y-\Delta am\,y \sin am\,y\,\Delta am\,x \sin am\,x}{1-k^2 \sin^2 am\,x \sin^2 am\,y}, \tag{45}$$

$$\Delta am\,(x+y) = \frac{\Delta am\,x\,\Delta am\,y-k^2 \cos am\,y \cos am\,x \sin am\,y \sin am\,x}{1-k^2 \sin^2 am\,x \sin^2 am\,y}. \tag{46}$$

These formulæ of addition may be obtained in various other well known forms, by using other combinations of the equations (16)—(21) with the equations (28)—(30).

ART. 10.—The elliptic function of the second species is defined by the equation

$$Z(x) = \int_0^x k^2 \sin^2 am\, x\, dx. \tag{47}$$

Observing that $$\sqrt{k} = \frac{\theta_{1,0}}{\theta_{0,0}} = \frac{1}{i}\,\frac{\theta'_{1,1}}{\theta_{0,1}},$$

we may write the equation (32) in the form

$$k^2 \sin^2 am\, x = \frac{\theta''_{0,1}(0)}{\theta_{0,1}(0)} - \frac{d}{dx}\left[\frac{\theta'_{0,1}(x)}{\theta_{0,1}(x)}\right],$$

whence, integrating, we obtain

$$Z(x) = x\,\frac{\theta''_{0,1}(0)}{\theta_{0,1}(0)} - \frac{\theta'_{0,1}(x)}{\theta_{0,1}(x)}, \tag{48}$$

which is Jacobi's expression for the elliptic function of the second species.

If x_1, x_2, x_3 are three arguments of which the sum is zero, we have

$$Z(x_1)+Z(x_2)+Z(x_3) = -\frac{\theta'_{0,1}(x_1)}{\theta_{0,1}(x_1)} - \frac{\theta'_{0,1}(x_2)}{\theta_{0,1}(x_2)} - \frac{\theta'_{0,1}(x_3)}{\theta_{0,1}(x_3)}.$$

The right-hand member of this equation may be transformed, by means of the formula (34), so as to contain only elliptic functions of the first species. We thus obtain

$$\begin{aligned} Z(x_1)+Z(x_2)+Z(x_3) &= \frac{\theta'_{1,1}(0)\,\theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3)}{\theta_{0,1}(0)\,\theta_{0,1}(x_1)\,\theta_{0,1}(x_2)\,\theta_{0,1}(x_3)} \\ &= k^2 \sin am\, x_1 \sin am\, x_2 \sin am\, x_3, \end{aligned} \tag{49}$$

which is the theorem of addition for elliptic functions of the second species.

ART. 11.—The elliptic function of the third species is defined by the equation

$$\Pi(x, y) = \int_0^x \frac{k^2 \sin am\, y \cos am\, y\, \Delta am\, y \sin^2 am\, x\, dx}{1 - k^2 \sin^2 am\, y \sin^2 am\, x}, \tag{50}$$

and is a function of two quantities, the argument x and the parameter y.

Dividing the equation (31) by $\theta^2_{0,1}(x)\,\theta^2_{0,1}(y)$, we find

$$\begin{aligned} 1 - \frac{\theta^2_{1,1}(y)\,\theta^2_{1,1}(x)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)} &= 1 - k^2 \sin^2 am\, y \sin^2 am\, x \\ &= \frac{\theta_{0,1}(x+y)\,\theta_{0,1}(x-y)\,\theta^2_{0,1}(0)}{\theta^2_{0,1}(y)\,\theta^2_{0,1}(x)}. \end{aligned}$$

Take the logarithm of each side; differentiate with respect to y, and then integrate with respect to x from 0 to x: we obtain

$$(51) \qquad \Pi(x, y) = x\frac{\theta'_{0,1}(y)}{\theta_{0,1}(y)} + \tfrac{1}{2}\log\frac{\theta_{0,1}(x-y)}{\theta_{0,1}(x+y)},$$

the important equation by which Jacobi has expressed the function $\Pi(x, y)$ containing two indeterminates by means of functions of one indeterminate. If x_1, x_2, x_3 are three arguments of which the sum is zero, we have

$$(52) \quad \Pi(x_1, y)+\Pi(x_2, y)+\Pi(x_3, y) = \tfrac{1}{2}\log\frac{\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)\,\theta_{0,1}(x_3-y)}{\theta_{0,1}(x_1+y)\,\theta_{0,1}(x_2+y)\,\theta_{0,1}(x_3+y)}.$$

The quantity after the logarithmic sign can be transformed (and that in various ways) by means of the formula (3), so as to contain only elliptic functions of the first species. Thus, attributing to the indeterminates and indices in the formula (3) the values

$$\begin{vmatrix} -y, & -x_1+y, & -x_2+y, & -x_3+y \\ 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \end{vmatrix},$$

we get

$$(53) \quad 2\theta_{0,1}(y)\,\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)\,\theta_{0,1}(x_3-y) = \theta_{0,1}(2y)\,\theta_{0,1}(x_1)\,\theta_{0,1}(x_2)\,\theta_{0,1}(x_3)+\theta_{0,0}(2y)\,\theta_{0,0}(x_1)\,\theta_{0,0}(x_2)\,\theta_{0,0}(x_3) +\theta_{1,1}(2y)\,\theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3)-\theta_{1,0}(2y)\,\theta_{1,0}(x_1)\,\theta_{1,0}(x_2)\,\theta_{1,0}(x_3).$$

In this equation change y into $-y$, and divide the first result by the second: the formula (52) becomes

$$(54) \quad \Pi(x_1, y)+\Pi(x_2, y)+\Pi(x_3, y) = \tfrac{1}{2}\log\frac{1+\Delta\,am\,2y\,[\Delta\,am\,x]-\cos am\,2y\,[\cos am\,x]+\sin am\,2y\,[\sin am\,x]}{1+\Delta\,am\,2y\,[\Delta\,am\,x]-\cos am\,2y\,[\cos am\,x]-\sin am\,2y\,[\sin am\,x]},$$

where $[\sin am\,x]$, $[\Delta\,am\,x]$, $[\cos am\,x]$ are written as abbreviations for $k^2 \sin am\,x_1 \sin am\,x_2 \sin am\,x_3$, $\frac{1}{k'^2}\,\Delta\,am\,x_1\,\Delta\,am\,x_2\,\Delta\,am\,x_3$, $\frac{k^2}{k'^2}\cos am\,x_1 \cos am\,x_2 \cos am\,x_3$. This is a very symmetrical form of the theorem of addition of functions of the third species: the equation of Legendre

$$(55) \quad \Pi(x_1, y)+\Pi(x_2, y)+\Pi(x_3, y) = \tfrac{1}{2}\log\frac{1+k^2\sin am\,y\sin am\,x_1\sin am\,x_2\sin am\,(x_3+y)}{1-k^2\sin am\,y\sin am\,x_1\sin am\,x_2\sin am\,(x_3-y)}$$

is, however, more convenient for computation. It may be obtained by attributing to the indeterminates and indices in the formula (3) the

values $\begin{vmatrix} x_1-y, & x_2-y, & x_3, & 0 \\ 0\ , & 0\ , & 0, & 0 \\ 1\ , & 1\ , & 1, & 1 \end{vmatrix}$, $\begin{vmatrix} x_1-y, & x_2-y, & x_3, & 0 \\ 1\ , & 1\ , & 1, & 1 \\ 1\ , & 1\ , & 1, & 1 \end{vmatrix}$,

and adding the results: we thus find

$$(56)\quad 2\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)\,\theta_{0,1}(x_3)\,\theta_{0,1}(0)$$
$$=\theta_{0,1}(x_1)\,\theta_{0,1}(x_2)\,\theta_{0,1}(x_3+y)\,\theta_{0,1}(y)+\theta_{1,1}(x_1)\,\theta_{1,1}(x_2)\,\theta_{1,1}(x_3+y)\,\theta_{1,1}(y).$$

Changing y into $-y$, dividing the first result by the second, and substituting the resulting expression of $\frac{\theta_{0,1}(x_1-y)\,\theta_{0,1}(x_2-y)}{\theta_{0,1}(x_1+y)\,\theta_{0,1}(x_2+y)}$ in the equation (52), we arrive at the formula (55).

Jacobi has given, in the *Fundamenta Nova*, two other expressions of this addition theorem, both of which are easily inferred from (52) by using appropriate particularisations of the formula (3).

On an Involution System of Circular Cubics, and description of the curve by points, when the double focus is on the curve.

By T. COTTERILL, M.A.

It follows from a theorem given by Chasles, that, if a system of circular cubics can pass through seven points, the tangents at each point sweep through equal angles; and consequently, if the system can break up in at least three ways into a circle and line, the circle described by the double focus or intersection of the tangents at the circular points and the conic of intersection of the tangents at any two of the other points are determined. The six intersections of four lines in a plane, and the focus of the inscribed parabola, afford an instance of such a system.

A particular case of this is given from the three points of a triangle (determined by a point on a circle and its intersections with a line), and the four centres of the circles touching its sides.

Any point on a circular cubic through seven such points determines another point on the curve, its confocal to the triangle, the line joining them being parallel to the real asymptote, so that a circular cubic is the locus of the real foci lying on parallel axes of conics inscribed to a triangle. The intersection of the tangents to the cubic at these

confocal points, is the confocal of the centre of the conic. In other words, the locus of the intersection of tangents to a circular cubic at the points in which it is cut by a line parallel to its real asymptote, is the confocal of the polar conic of the real point at infinity.

The tangents at the four centres are parallel to the real asymptote. The tangents at the angles of the triangle and the real asymptote meet at the remaining intersection of the cubic and circle circumscribing the triangle, the asymptote being easily drawn, as it is the tangent in a given direction to the three-cusped hypocycloid touching the six lines through the centres, one pair of which is always real.

The double focus is on the same circle, diametrically opposite the last mentioned point.

The single foci are known to be in fours on the polar circles of the triangles formed from the centres, and found from the principles laid down in Dr. Salmon's "Higher Geometry" by means of double lines of Involution.

The cubics break up in six ways into a circle and line, one such system being orthogonal to another. The common foci (which coalesce in fours) of an orthogonal pair are the nodes of each, considered as a cubic.

When the asymptote is perpendicular to a side of the triangle, and the double focus at its opposite angle and therefore on the curve, the genesis of the curve is so easy, and its properties, both metric and graphic, become so simple, as to entitle it to hold the rank amongst cubics somewhat similar to that of the circle amongst conics.

The locus of the intersections of circles through two fixed points with their diameters through another fixed point, is a circular cubic having the last mentioned point for its double focus.

Hence it is also the locus of the points of contact of tangents through the fixed point to the orthogonal system of circles.

The envelope of the line joining two corresponding points on a 3rd order cubic, considered as a Hessian, is called by Prof. Cremona a *Cayleyan*, in honour of the discoverer of the remarkable relations of the two curves.

In the circular cubic in question, the distance between one set of pairs of corresponding points (the circular points at infinity being one pair) is bisected by the line of centres of the circles, and corresponding tangents to the Cayleyan are the orthogonal lines through any point on the Hessian, cutting the axis parallel to the line of centres through the double focus, in points equidistant from it.

The tangent at a point of the Hessian, and point of contact of a tangent to the Cayleyan, are found from the property given by Prof. Cayley; viz., that corresponding points are harmonicals to corresponding tangents.

According as the axis of the coaxial circles is real, imaginary, or evanescent, the Hessian is complex, simplex, or orthogonally crunodal, and the Cayleyan is complex, simplex, or a parabola, and the limiting point of the evanescent axis.

The Hessian is the locus of the points of contact of pairs of conics with fixed foci, the tangent and normal at the point of contact being corresponding tangents to the Cayleyan.

It is also the locus of the foci of conics touching four lines, the Cayleyan being the envelope of their axes.

October 15, 1866.

Special Meeting held at University College (Professor De Morgan in the Chair).

Professor Hirst read a letter from Sir John Lubbock with reference to a present, which he intended to make to the Society, of a considerable portion of the Mathematical Library of the late Sir John Lubbock.

It was resolved, That the warmest thanks of the Society be returned to Sir John Lubbock for his very generous and valuable gift.

A new scheme of rules, which had been drawn up by Professors Hirst and Sylvester at the request of the Council, was then read by Professor Hirst. These rules were passed with certain alterations.

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